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Master Degree Thesis

Quantization of gravitational radiation over a Schwarzschild spacetime and its semiclassical backreaction: A model of black hole evaporation

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To those that are not with us anymore

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When you bow deeply to the universe,
it bows back;
when you call out the name of God,
it echoes inside you.

- Morihei Ueshiba, The Art of Peace

Abstract

The construction of a quantum theory of gravity has fascinated generations of physicist and mathematicians for over a century. Unfortunately, despite the many partially successful attempts, no universally accepted solution has been obtained yet. On the other hand, also black hole physics has been of particular interest among the literature, since it naturally provides a forge to test and understand the role of quantum mechanics in the description of gravitational interactions.

In this thesis we apply the techniques provided by both quantum field theory on curved spacetimes and semiclassical gravity to the case of a spherically symmetric static black hole, namely when the background is described by the Schwarzschild metric. When a black hole evaporates, its area gradually decreases as the consequence of a loss of mass. The rate of change of area is controlled by the Raychaudhuri's equation for a congruence of radial null outgoing geodesics, which actually describes the geometrical structure of the event horizon. We consider the role of quantum gravitational radiation in black hole evaporation, estimating the backreaction of the perturbation field on the background spacetime, by means of a semiclassical contribution to the Raychaudhuri's equation.

To achieve this result and after a brief introduction, we review the theory of linearized gravity, focusing on its quantization in the algebraic framework, on the construction of Hadamard quantum states and on the computation of the expectation values of observables. Then, we adopt a perturbative approach to the Raychaudhuri's equation, focusing on the effects of backreaction along the event horizon, which can be studied in terms of the expectation value of the stress-energy tensor of quantized gravitational radiation. To this extent, we account for the divergences of the two-point correlation functions of states, regularizing the local products of perturbation fields by means of the point-splitting procedure, which is ensured by the universality of the singularities of Hadamard states.

Under some physical assumption on the state and by some explicit computations, we argue that for a radial null outgoing congruence, the trace anomaly of the renormalized stress-energy tensor contributes positively to the Raychaudhuri's equation. Such a result highlights that (initially static) spherically symmetric black holes *can* evaporate by emission of gravitational radiation, which is due to the presence of backreaction of the quantum field on the spacetime, and which can be related, at large distance, to the presence of gravitational Hawking radiation.

Sommario

La costruzione di una teoria quantistica della gravità ha affascinato generazioni di fisici e matematici per più di un secolo. Sfortunatamente, nonostante diversi tentativi parzialmente riusciti, non è stata ancora trovata alcuna soluzione universalmente accettata. D'altro canto, anche la fisica dei buchi neri ha suscitato particolare interesse nella letteratura, dal momento che fornisce naturalmente una fucina per testare e comprendere il ruolo della meccanica quantistica nella descrizione delle interazioni gravitazionali.

In questa tesi applichiamo le tecniche proprie della teoria dei campi su spazitempi curvi e della gravità semiclassica al caso di un buco nero statico e sfericamente simmetrico, cioè quando lo spaziotempo di background è descritto da una metrica di Schwarzschild. Quando un buco nero evapora, la sua area decresce gradualmente come conseguenza di una perdita di massa. Il tasso di variazione dell'area è controllato dall'equazione di Raychaudhuri per una congruenza di geodetiche radiali di tipo luce e uscenti, che descrive la struttura geometrica dell'orizzonte degli eventi. In particolare, consideriamo il ruolo della radiazione gravitazionale quantistica nell'evaporazione di un buco nero, stimando la controreazione della perturbazione sullo spaziotempo di background, in termine di un contributo semiclassico all'equazione di Raychaudhuri.

Per ottenere questo risultato e dopo una breve introduzione, riassumiamo la teoria della gravità linearizzata, concentrandoci sulla sua quantizzazione nel contesto algebrico, sulla costruzione degli stati quantistici di Hadamard e sul calcolo del valore di aspettazione delle osservabili. In virtù di questo, analizziamo l'equazione di Raychaudhuri da un punto di vista perturbativo, focalizzando la nostra attenzione sugli effetti prodotti dalla controreazione lungo l'orizzonte degli eventi, i quali possono essere studiati in termini del valore di aspettazione del tensore energia-impulso della radiazione gravitazionale quantistica. A tal fine, è necessario tener conto delle divergenze delle funzioni di correlazione a due punti degli stati, regolarizzando i prodotti locali delle perturbazioni attraverso la procedura di point-splitting, la quale è assicurata dall'universalità delle singolarità degli stati di Hadamard.

Sotto qualche assunzione fisica sulla stato e attraverso qualche calcolo esplicito, dimostriamo che per una congruenza di geodetiche radiali di tipo luce e uscenti, l'anomalia di traccia del tensore energia-impulso rinormalizzato contribuisce positivamente all'equazione di Raychaudhuri. Un contributo di questo tipo sottolinea che buchi neri a simmetria sferica inizialmente statici *possono* evaporare per emissione di radiazione gravitazionale, la quale è dovuta all'influenza della controreazione del campo quantistico sullo spaziotempo, e che può essere legata, a grandi distanze, alla presenza di radiazione di Hawking gravitazionale.

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Introduction

In this thesis we consider a new model of black hole evaporation, by the computation of the backreaction of quantum gravitational radiation on a static and spherically symmetric background, employing the techniques of both quantum field theory on curved spacetimes and semiclassical gravity.

From a classical point of view, general relativity predicts black holes to have an event horizon, namely a surface which can trap the geodesics within its internal region such that nothing can escape from it. However, this well known consequence of the Einstein theory does not account for quantum mechanics. A first insight on this has been given by Hawking in the 70s [25], studying the presence of quantum mechanical effects given by the combination of pair production mechanism and the quantum tunnelling across the event horizon. Asymptotically this leads to the production of a flux of particles from the black hole, whose frequency spectrum follows the Bose-Einstein statistics. Actually, the power of this radiation can be computed and assumed to be related to the rate of change of mass of the black hole, which as a consequence of this escape of energy starts evaporating (i.e losing its mass). Here the mechanism is assumed to be adiabatic, that is the spacetime is required to be described by the Schwarzschild metric, with a different value of the mass as time changes. Moreover, and despite its original description in terms of an emission of photons, it has been shown that other particles can actually contribute to the evaporation process [39, 40, 41]. However, all these considerations start from the assumption that the evaporation actually occurs, neglecting the deep modifications brought to the spacetime by such a process, while actually limiting the discussion to a global point of view in regard to the flux of radiation emitted by the black hole [8].

In this thesis we consider a local description of black hole evaporation, showing that both gravitational and scalar Hawking radiation are predicted by quantum field theory on curved spacetimes and semiclassical gravity, actually as a consequence of the backreaction of quantum fields on a background spacetime, endowed with a Schwarzschild (namely static and spherically symmetric) metric.

The unification of general relativity with the description of microscopic effects given by quantum mechanics is far from being completely understood. However, quantum field theory on curved spacetime works as a first, although rich, approximation to the complete solution, by studying the propagation of quantum fields on a curved and fixed spacetime background. In this thesis, we consider the quantization of linearized gravity, thus building a model for the propagation of quantum gravitational radiation around a spherically symmetric black hole. Then we consider the semiclassical Einstein equation, as a second step towards a quantum description of the gravitational interaction, assuming that the background may actually change under the influence of the propagating quantum field, controlled by the expectation value of its energy-tensor and leading to the so-called backreaction [22, 47, 23, 33]. To exploit the global relation between the semiclassical Einstein equation and the production of Hawking radiation from black holes, we adopt a different point of view from [25]. We study the influence of the backreaction on the geometrical properties of a congruence, i.e a bundle, of radial null outgoing geodesics, which describe the behaviour of lights rays around the black hole, thus encoding the geometrical structure of the horizon. The presence of evaporation can be established studying the behaviour of θ, the expansion parameter, which is defined as the rate of cross-sectional area growth of the congruence, while actually being related to the flux of outgoing geodesics (i.e energy and particles) across the horizon. The dynamical evolution of the expansion is described by the Raychaudhuri's equation [48, 49], which we study adopting a perturbative point of view, thus using quantum field theory to investigate the contributions of quantum gravitational perturbations, which are modelled by an external field propagating on the curved background [1]. By recalling the quantization of linearized gravity [18, 24], we select a suitable state [31, 22, 1] to compute the backreaction of the field in terms of the semiclassical Einstein equation [33, 47]

$$G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle \ . \tag{1}$$

which provides a nice way to link the information carried by the quantum theory, encoded in the stress-energy tensor $T_{\mu\nu}$, with the geometry of the spacetime driven by the Einstein tensor $G_{\mu\nu}$. To achieve such a result, we need to renormalize the expectation value of $T_{\mu\nu}$, thus getting an expression for the quantum stress-energy tensor which makes finite the contribution of the quantum fluctuations of the spacetime, while allowing us to write (1). Using the semiclassical Einstein equation, we can control the rate of change of the expansion parameter on the horizon $\mathcal{H}$ by

$$\frac{d\theta}{d\lambda}\Big|_{\mathcal{H}} = -8\pi G \langle T_{\mu\nu} \rangle k^\mu k^\nu \ , \tag{2}$$

with k^μ the vector field tangent to the geodesics congruence. However, the quantum stress-energy tensor shows a trace anomaly, which is produced by the simultaneous requirement of $T_{\mu\nu}$ being finite and covariantly conserved (thus obeying to the principle of conservation of energy) and which depends only from the choice of the model and of the spacetime background [1, 10, 6]. In our work we show that this anomalous term, under some physical assumption on the quantum state, is the only positive contribution in (2), thus giving

$$\frac{d\theta}{d\lambda}\Big|_{\mathcal{H}} > 0 \ .$$

We argue that for a initially static (i.e stable in the past) and spherically symmetric black hole, the trace anomaly contribution leads to a growing flux of outgoing geodesics through the event horizon. In this way, we see that the quantum modifications induced on the spacetime by the gravitational perturbation act as source of violation of the trapping capability of the horizon, allowing for the geodesics to escape from it and thus producing flux of gravitational energy to the future infinity, being responsible for the loss of mass of the black hole and its actual evaporation.

Finally, we show the usual interpretation of Hawking radiation by relating this flux of energy to the presence of radiation at large distance, to which we associate and compute the luminosity, actually recovering the result previously found by Hawking [25]

$$\mathcal{L} = \frac{\alpha}{\pi M^2} \ .$$

It is worth to note that this procedure can be applied to any field theory, starting from a different choice of the quantum stress-energy tensor and studying the evaporation in terms of its trace anomaly contribution [10, 6].

The contents of this work

Part I In the first part of this work we consider the geometrical notions behind the evaporation process of a stationary spherically symmetric black hole. We start with a review of the fundamental contents of general relativity, by giving further mathematical details regarding the causal structure on curved spacetimes. Here we discuss the notion of globally hyperbolic spacetimes, which are the natural framework to describe any classical or quantum field theory. Finally, we recall the description of static and spherically symmetric black holes in terms of the Schwarzschild metric. As it is, this choice of the spacetime requires additional care, being ill-defined on the event horizon. Indeed, we shall pay attention to its Kruskal extension, aimed to achieved regularity and a more complete description (see [48, 49, 15] as the main references on these topics).

After this brief introduction we consider the evaporation in terms of a positive flux of gravitational energy across the event horizon. To this extent, we review the geometrical notion of geodesic congruence [49, 8]. We consider the Raychaudhuri's equation, which accounts for the deformation of the congruence due to spacetime curvature, by investigating the relation between the flux of geodesics and the rate of change of the cross-sectional area of the bundle [48, 9]. On a perturbation-free Kruskal background, we will explicitly argue that the Raychaudhuri's equation correctly reproduces the stability of the event horizon and the absence of evaporation. Finally, we consider a linear expansion of the background metric, investigating the perturbation theory behind the Raychaudhuri's equation and lying the foundation for the description of the evaporation in terms of linearized quantum gravitational radiation [28].

Part II In the second part of this work we review both the classical and quantum aspects of linearized gravity on curved spacetime, being established that linear perturbations give a satisfactory description of the evaporation by a second order correction to the derivative of the flux of geodesics across the horizon. The first chapter of this part is devoted to a classical discussion, starting from the linearization of the Einstein tensor, which leads to the equation of motion (EoM) of a gravitational wave on a general curved background [18, 5, 24]. Therefore, we exploit the gauge symmetry carried by linearized diffeomorphisms, requiring that the perturbation field satisfies the de Donder gauge condition. By this way, the equation of motions can be put in a normally hyperbolic form, thus ensuring the existence of the advanced and retarded propagators as the fundamental solutions of the field equations. This property also guarantees the existence and uniqueness of the solutions of the

Cauchy problem associated to the EoM, once equipped with a suitable set of initial data [4, 3, 24].

In order to achieve a fundamental quantum description of gravitational evaporation, in the second chapter of this part we review the algebraic quantization of linearized gravity on generally curved backgrounds. On a curved spacetime, there are actually many problems related to the usual approach to the quantization in terms of the construction and annihilation operators. To this extent, one need to choose a suitable local coordinate frame and a local time function (if there exists) to define the Fourier transform and thus compute the positive and negative frequencies coefficients. Therefore, the former can be used to build the one-particle structure on a suitable Hilbert space, achieving the second quantization while promoting the coefficients of the Fourier transform to operator on the Fock space. The essence of this construction can be understood as an attempt to give the canonical commutation relations while simultaneously choosing the representation of fields on the Fock space, which however leads to a non-covariant formulation. The solution to this problem is given by the algebraic approach, which splits the construction of the algebra of fields from the choice of the representation on the Hilbert space, thus providing a quantization scheme which holds for any choice of the spacetime background.

Contrary to the Klein-Gordon theory, reviewed in the first appendix, the quantization of linearized gravity requires the additional care reserved to gauge theories. Indeed, the quantization of the symplectic structure is bounded to the existence of the fundamental propagators, or similarly of the conjugated momenta, which has been ensured under a suitable gauge-fixing process [18, 1]. In order to resume the freedom lost by the choice of the de Donder gauge, we discuss two different approaches. On one hand, we recover the gauge symmetry by identifying the observables within the quotient of the space of solutions of the EoM with respect to that of pure gauge fields, actually reconstructing the freedom while being on-shell [18, 24]. This procedure allows us to get a gauge-invariant formulation of the notion of observable, avoiding the introduction of other fields terms within the action and by simply translating the gauge condition to restrict the choice of possible test functions. However, we argue that this way reveals to be unsatisfactory in order to construct the coinciding-point limit of observables made of products of fields, leading to undesired conditions which cannot be easily absorbed by the test functions. To prevent this possible limitation, we consider the use of ghost fields to recover the gauge symmetry, which can be achieved before employing the EoM, by means of the action of a BRS operator, whose cohomology defines the space of gauge-invariant observables [16, 17, 38]. In both the cases, the quantization can be achieved by promoting the fields to generator of a $*$-algebra, whose structure is covariantly fixed by the canonical commutation relations, independently from the choice of the representation on the Fock space [31, 22].

Once that the algebraic structure of the theory is understood, we need to compare this mathematical framework with the physical reality, which passes by a process of measure, thus requiring the notion of quantum state. Therefore, we define the states as functional acting on the algebra generated by fields. We discuss the definition of quasi-free states, whose behaviour is completely fixed by the two-point correlation function through the Wick theorem. In particular, we consider the restriction to Hadamard quasi-free states, as the only ones to be physically admissible, mimicking the divergent behaviour of the vacuum state on the Minkowski spacetime and thus allowing the recover of the one-particle structure of the theory. Those divergences are universally described by the Hadamard parametrix (i.e independently from the choice of the state itself), which can be suitably subtracted to regularize the expectation value of observables. To this extent, we conclude this chapter by recalling the point-splitting procedure, used to regularize the Wick monomials built by taking the product of fields in the coinciding-point limit (see [31, 22, 23, 1] as the main references).

Part III In the last part of this work we use the framework of algebraic quantum field theory on curved spacetimes to describe the gravitational evaporation on a stationary spherically symmetric background. By recalling the results of the first part, we consider the description of the evaporation in terms of the perturbed Raychaudhuri's equation, whose contributions now depend from the quantum field that describes the gravitational perturbation. To estimate the backreaction of the quantum gravitational radiation, we employ the semiclassical Einstein equation [22, 47]. However, since the stress-energy tensor is quadratic in the fields and their derivatives, its definition as a functional derivative of the action leads to a divergent expectation value. For this reason, we pursue its renormalization, by subtracting the Hadamard divergent contribution to obtain a well-defined prescription, while exploiting the freedom of the point-splitting procedure to preserve the covariant conservation property. Finally, we get a quantum prescription which also obeys to the principle of conservation of energy [1, 23, 22]. However, this renormalization technique modifies the trace of the tensor itself, thus producing an anomalous term, which only depends from the choice of the background spacetime [1]. By a suitable choice of the state (which is required to satisfy the same properties of the Unruh one of the Klein-Gordon theory), we argue that the trace anomaly actually drives the evaporation of the black hole, bringing a positive contribution to the Raychaudhuri's equation, and thus leading to a growth of the flux of outgoing gravitational energy across the event horizon. Finally, we associate this flux of energy to the presence of Hawking radiation at large distance from the black hole, of which we compute the luminosity.

Part I

The geometrical interpretation of black hole evaporation

Chapter 1

Geodesics congruence

1.1 Introduction

The main purpose of this thesis is to study the contribution of quantum gravitational radiation to black hole evaporation. Usually, evaporation phenomena are studied by means of Hawking radiation, namely as a positive flux of energy computed at asymptotic distances from the black hole [8]. We present an alternative point of view, by studying the effect of gravitational radiation on the structure of the event horizon. In these terms, the evaporation of a black hole can be understood by the presence of gravitational outgoing energy across the horizon itself. Before discussing how quantum effects produced by the gravitational field may influence this process, we need to investigate the classical details behind the relation between the curvature of spacetime and the shape of family of a geodesic, namely a congruence.

In this chapter we review the basic notion of differential geometry and general relativity, which will play a pivotal role in the entire work. Given all the geometrical details, we shall discuss the notion of geodesic congruence and its relation with black hole evaporation, whose geometrical property will be described by means of the Raychaudhuri's equation. We will end this introduction by giving a first outlook towards a perturbative analysis of the Raychaudhuri's equation, which will motivate an analysis in the framework of linearized gravity on curved spacetimes.

1.2 Geometrical and physical preliminaries

We begin our discussion with a short review of all the geometrical and physical notions, which will be often used during our thesis. Most of these definitions are quite standard among the literature and they can be found for instance in [48, 9, 37].

The purpose of general relativity is to understand the relation between the gravitational field and the presence of matter or energy, by means of two fundamental principles.

1. A body which is subject to the action of a gravitation field experiences a free-falling motion along the spacetime, whose trajectory is described by a *geodesic*.

2. The presence of a gravitational field can be understood geometrically as the effect of the curvature of *spacetime*, which is generated by the presence of

either matter or energy.

We now introduce all the geometrical notions which allow a precise mathematical description of the physical content of these principles.

We call *spacetime* a couple $(\mathcal{M}, \boldsymbol{g})$, with $\mathcal{M}$ being a smooth manifold and $\boldsymbol{g}$ a Lorentzian metric of signature $(-,+,+,+)$.

We associate to the manifold $\mathcal{M}$ a *tangent bundle* $T\mathcal{M}$ and its dual $T\mathcal{M}^*$, the *cotangent bundle*, which are respectively the space of vector fields and dual vector (one-forms) fields on $\mathcal{M}$.

To fix the notation, we briefly consider a vector field $\boldsymbol{V} \in T\mathcal{M}$ and a 1-forms field $\boldsymbol{\omega} \in T\mathcal{M}^*$. Both can be explicitly written in terms of a basis, once that a suitable local coordinate frame $\{x^a\}$ has been chosen, thus giving

$$\boldsymbol{V} = V^a(x)\partial_a, \quad \boldsymbol{\omega} = \omega_a(x)dx^a \ .$$

The notion of vectors and 1-forms allow us to give the definition of a generic (h, l) tensor field, as a pointwise multilinear map

$$\boldsymbol{T} : (T\mathcal{M}^*)^h \times (T\mathcal{M})^l \longrightarrow C^\infty(\mathcal{M}) \ ,$$

which is completely fixed by its action on the elements of the basis, i.e by its components $T^{i_1 \dots i_h}{}_{j_1 \dots j_l}$.

When dealing with curved spacetimes, also the idea of derivation needs to be revised. For instance, taking the derivative of $\boldsymbol{V}$, requires the subtraction of two vectors, of which the former needs to be parallel transported on the latter, starting from a different point of the manifold. However, on general curved spacetimes there is no translational invariance, hence, different paths on $\mathcal{M}$ leads to different results of the derivative. The dependence from the transport direction is precisely described by the covariant derivative ∇, whose action, when viewed as a differential operator, depends on the geometrical nature of its argument. Indeed, given two vector fields $\boldsymbol{V}, \boldsymbol{U}$, the covariant derivative of $\boldsymbol{V}$ along the $\boldsymbol{U}$-direction reads

$$\nabla_{\boldsymbol{U}} \boldsymbol{V} = U^b \nabla_b V^a \partial_a = U^b (\partial_b V^a + \Gamma^a{}_{bc} V^c) \partial_a \ , \tag{1.1}$$

with $\nabla_a \partial_b = \Gamma^c{}_{ab} \partial_a$. Hence, we say that $\boldsymbol{V}$ is *parallel transported* along $\boldsymbol{U}$ if

$$\nabla_{\boldsymbol{U}} \boldsymbol{V} = 0 \ . \tag{1.2}$$

The coefficients of the connection $\Gamma^a{}_{bc}$ and the metric $\boldsymbol{g}$ are related by the *Levi-Civita theorem*. Indeed, given the metric-compatibility condition

$$\nabla_a g_{bc} = 0 \ ,$$

there exists an unique symmetric connection, i.e $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$, satisfying

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right) \ . \tag{1.3}$$

In such a case $\Gamma^a{}_{bc}$ are usually referred to as *Christoffel symbols*.

The relation between the curvature of the spacetime and the error made by choosing two different paths of derivation, is described by the Riemann tensor. Indeed, adopting the convention of [48], given a vector field $\boldsymbol{V}$, the *Riemann tensor* is defined as

$$R_{abc}{}^{d}V^{c} \doteq -(\nabla_a\nabla_b - \nabla_b\nabla_a)V^{c} \ .$$

By exploiting the definition of the covariant derivative (1.1), we get the components of the Riemann tensor, expressed in terms of the Christoffel symbols (1.3)

$$R_{abc}{}^{d} \doteq \partial_b\Gamma^{d}{}_{ac} - \partial_a\Gamma^{d}{}_{bc} + \Gamma^{e}{}_{ac}\Gamma^{d}{}_{eb} - \Gamma^{e}{}_{bc}\Gamma^{d}{}_{ea} \ ,$$

which leads to

$$R_{ab} \doteq R_{acb}{}^{c}, \quad R \doteq g^{ab}R_{ab} \ .$$

respectively called the *Ricci tensor* and *Ricci curvature*. Then, the *Einstein tensor* can be defined as

$$G_{ab} \doteq R_{ab} - \frac{1}{2}Rg_{ab} \ ,$$

which satisfies the Bianchi identity $\nabla^a G_{ab} = 0$.

The stress-energy tensor T_{ab} is a (0,2) symmetric, covariantly conserved tensor, i.e $\nabla^a T_{ab} = 0$, which plays a pivotal role in general relativity. Indeed, it allows us to implement principle 2, relating the presence of matter and energy with the shape of spacetime curvature, by means of the Einstein field equation

$$G_{ab} = 8\pi G T_{ab} \ , \tag{1.4}$$

with G the Newtonian constant of gravitation. By multiplication of both members of (1.4), for the inverse metric g^{ab}, we can equivalently state that

$$R_{ab} = 8\pi G\left(T_{ab} - \frac{1}{2}Tg_{ab}\right) \ . \tag{1.5}$$

with $T \doteq g^{ab}T_{ab}$. Without matter or energy, (1.4) leads to the vacuum Einstein field equation

$$G_{ab} = 0 \ . \tag{1.6}$$

As stressed before, the Einstein equation describes completely the dynamics of the spacetime in terms of its solution $\boldsymbol{g}$. Once the geometry of the spacetime is known, the trajectory of a free falling body can be described by means of the notion of geodesic.

We call *geodesic* a curve $\boldsymbol{\gamma}$, whose tangent vector field $\boldsymbol{k}$ is parallel transported along the curve itself. From definition (1.2), it follows that

$$k^a\nabla_a k^b = 0 \ , \tag{1.7}$$

which is called the *geodesics equation*. This expression is actually well-posed, despite $\boldsymbol{k}$ being a vector field defined only on $\boldsymbol{\gamma}$. Indeed, prescription (1.7) involves only the projection of the covariant derivative along the direction of $\boldsymbol{k}$, evaluated along $\boldsymbol{\gamma}$.

The geometrical requirement brought by (1.7) allows the interpretation of geodesics as those curves, straight with respect to the spacetime curvature, that actually describe the natural behaviour of a free falling body, thus implementing principle 1 of general relativity.

Before entering all the mathematical details, we briefly discuss the physical meaning behind the notion of *causality*. Let us consider two events s and s', connected by a curve $\boldsymbol{\gamma}$ with tangent vector field $\boldsymbol{V}$. The causal relation between s and s' can be understood by means of the *interval*, which is defined as

$$\Delta s^2 \doteq g_{ab} V^a V^b \ . \tag{1.8}$$

We say that s and s' are *timelike*, *null* or *spacelike separated*, if Δs^2 is respectively less, equal or greater than zero for every point of $\boldsymbol{\gamma}$ between s and s'.[1] This distinction has a deep physical consequence: if two events are spacelike separated, then there can be no cause-and-effect relation between them. Indeed, any spacelike interaction between s and s' would require a superluminal transmission, which is actually forbidden by special and general relativity. For this reason, two events which are either timelike or null separated are said to be *causal*. Finally, we note that, since (1.8) is a scalar quantity, the separation between two events does not depend upon the choice of the reference frame.

We can extend the the previous classification from events to curves, which will play a fundamental role in the description of the causal structure of the spacetime. Let $\boldsymbol{\gamma}$ be a curve, to which we associate a tangent vector field $\boldsymbol{k}$. We say that $\boldsymbol{\gamma}$ is *timelike*, *null* or *spacelike*, if $g_{ab}k^a k^b$ is respectively less, equal or greater than zero, for *every* point of the spacetime. Moreover, a curve is said *causal* if it is nowhere spacelike.

In order to gain a simple distinction between past and future within the spacetime $(\mathcal{M}, \boldsymbol{g})$, we require $\mathcal{M}$ to be *time-orientable* by assuming the existence of a global timelike vector field $\boldsymbol{t}$ (which is highly non-unique). By means of $\boldsymbol{t}$, for every point of the spacetime we can describe the light cone associated to a timelike curve $\boldsymbol{\gamma}$. We consider the following definition [48, Ch. 8], [4].

1. A timelike curve $\boldsymbol{\gamma}$, with tangent vector field $\boldsymbol{k}$, is said *future* or *past directed* if $g_{ab}k^a k^b$ is respectively everywhere greater or less than zero. Allowing for the equality, we trivially extend this definition to causal curves.

2. Let I, J two subsets of $\mathcal{M}$ such that $I \subsetneq J$. A geodesic $\boldsymbol{\gamma}$, defined in I, is said *inextensible* if does not exist any other geodesic $\widetilde{\boldsymbol{\gamma}}$, defined in J, such that $\boldsymbol{\gamma} = \widetilde{\boldsymbol{\gamma}}|_I$.

3. A subset $\Sigma \subset \mathcal{M}$ is called *achronal* if each timelike curve in $(\mathcal{M}, \boldsymbol{g})$ intersects Σ at most once.

4. A subset $\Sigma \subset \mathcal{M}$ is called *timelike*, *null* or *spacelike*, if its points are respectively all timelike, null or spacelike separated.

5. For any subset $\Sigma \subset \mathcal{M}$, we call *future domain of dependence* $D^+_{\mathcal{M}}(\Sigma)$ the collection of all point $q \in \mathcal{M}$ such that every past inextensible causal curve passing through q intersects Σ. Similarly, we can define the *past domain of dependence* $D^-_{\mathcal{M}}(\Sigma)$. We simply call *domain of dependence* the set $D_{\mathcal{M}}(\Sigma) \doteq D^+_{\mathcal{M}}(\Sigma) \cup D^-_{\mathcal{M}}(\Sigma)$.

[1] The possibility to have $\Delta s^2 < 0$ comes from the fact that spacetimes are endowed with Lorentzian metrics, which have indefinite signature.

6. For any subset $\Sigma \subset \mathcal{M}$, we call *causal future* $J^+_{\mathcal{M}}(\Sigma)$ the collection of all point $q \in \mathcal{M}$ such that there exists a future directed causal curve $\gamma : I \to \mathcal{M}$, for which $\gamma(0) = p$ and $\gamma(1) = q$, with $p \in \Sigma$. Similarly, we can define the *causal past* $J^-_{\mathcal{M}}(\Sigma)$. Finally, we simply denote $J_{\mathcal{M}}(\Sigma) \doteq J^+_{\mathcal{M}}(\Sigma) \cup J^-_{\mathcal{M}}(\Sigma)$

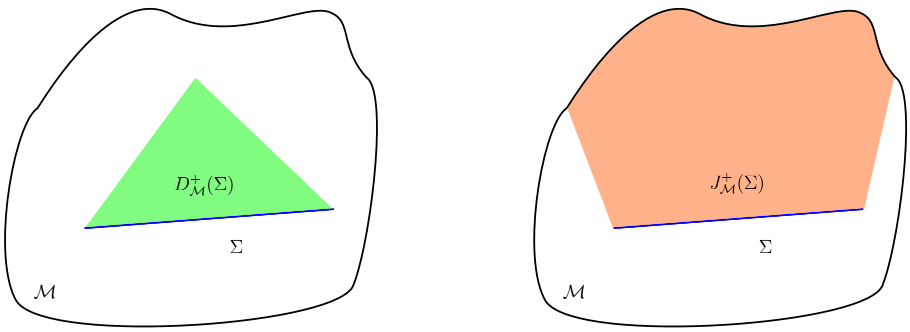
Intuitively, the causal future of a surface Σ contains all the points of the spacetime which are causally related to any point of Σ, i.e the union of the light cones generated by every point of Σ. On the other hand, the domain of dependence contains only those points which are causally related to every point of Σ. We sketch an example of $D^+_{\mathcal{M}}(\Sigma)$ and $J^+_{\mathcal{M}}(\Sigma)$ in figure 1.1, to clarify this distinction.

Finally we introduce two definitions that will be very useful to construct an initial value problem on $(\mathcal{M}, g)$.

1. Using the previous definitions, we call *Cauchy surface* any achronal subset $\Sigma \in \mathcal{M}$, such that $D_{\mathcal{M}}(\Sigma) = \mathcal{M}$.

2. A time-oriented spacetime $(\mathcal{M}, \boldsymbol{g})$ is said *globally hyperbolic* if and only if admits at least one Cauchy surface.

In the following chapters, we will exploit these geometrical notions to set up an initial value problem for a quantum field on a curved background. Indeed, the choice of a spacelike Cauchy surface will provide the idea of a surface with "constant time", with respect to a set of initial conditions can be fixed. By this way, it it will be possible to ensure the existence and uniqueness of solutions to the field equation, by also allowing the construction of the propagators of the theory.



(a) The future domain of dependence of Σ (b) The causal future of Σ

Figure 1.1: Causal structure

1.3 From the Schwarzschild solution to the Kruskal extension

In this section we briefly review the case of a static spherically symmetric black hole, described by the Schwarzschild metric and by its Kruskal extension.

Let us consider a spacetime $(\mathcal{M}, \boldsymbol{g})$, which satisfies the vacuum Einstein equation (1.6). Given $\boldsymbol{T} = 0$ then, (1.5) gives

$$R_{ab} = 0 \ . \tag{1.9}$$

This leads to a set of second order partial differential equations with the metric $\boldsymbol{g}$ as a solution, which a priori could be difficult to solve.

The situation can be definitely simplified, provided some ansatz on $\boldsymbol{g}$. Indeed, *Birkhoff's theorem* states that, for a spherically symmetric spacetime, (1.9) has unique solution represented by the Schwarzschild metric [48], which reads

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \ . \tag{1.10}$$

This metric is a solution of the vacuum Einstein equation (1.9) for $R_s < r < +\infty$. Actually, it describes a spacetime with a static spherically symmetric black-hole of mass M. The length $R_s = 2MG$ is called Schwarzschild radius, and it identifies an hypersurface $\mathcal{H}$ known as the *event horizon*.

When dealing with (1.10), it becomes clear that $\boldsymbol{g}$ is not well defined on $\mathcal{H}$, since it shows a singularity for $r \to R_s$. We discuss now an extension of the Schwarzschild spacetime, which removes the singularity on the event horizon and leads to a spacetime well-defined on $\mathcal{H}$ and beyond, being regular on $0 < r < +\infty$. This discussion will motivate the following distinction. On one hand, the event horizon represents a *coordinate singularity*, since it can be removed by a suitable coordinate transformation. From a physical point of view, this means that an observer would take a finite proper time in order to reach $\mathcal{H}$ from the outside [9, Ch. 5]. On the other hand, the point $r = 0$ represents a *physical singularity*, which cannot be removed in any way.

For the rest of this section we discuss how to bypass the coordinate singularity in $\mathcal{H}$, showing that the Schwarzschild spacetime can be extended to the more complete solution called the *Kruskal spacetime*. Immediately, we can observe that the angular part of the Schwarzschild metric (1.10) is simply the metric of a 2-sphere, which is well-behavioured on $\mathcal{H}$. Hence, we consider the two-dimensional restriction of $\boldsymbol{g}$ with coordinates (t, r), being the one that actually suffers under the limit $r \to R_s$

$$ds^2_{2D} = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 \ . \tag{1.11}$$

We define the *tortoise coordinate* r_*

$$r_* \doteq r + R_s \log(r/R_s - 1) \ , \tag{1.12}$$

which satisfies

$$\frac{dr_*}{dr} = \left(1 - \frac{R_s}{r}\right)^{-1} \ . \tag{1.13}$$

The *null coordinates* u, v are given by

$$\begin{aligned} u &\doteq t - r_* \ , \\ v &\doteq t + r_* \ . \end{aligned} \tag{1.14}$$

Using (1.14) together with (1.12), we can rewrite the metric (1.11) as

$$d\hat{s}^2_{2D} = -\left(1 - \frac{R_s}{r}\right) dudv \ . \tag{1.15}$$

Starting from prescription (1.14), we can define the *Kruskal null coordinates* U, V

$$U \doteq -e^{-u/2R_s} \ , \qquad V \doteq e^{v/2R_s} \ , \tag{1.16}$$

which, by substitution into (1.15), give

$$d\tilde{s}^2_{2D} = -\frac{4R_s^3 e^{-r/R_s}}{r} \, dU dV \ , \tag{1.17}$$

with the radial coordinate defined implicitly by

$$UV = \left(1 - \frac{r}{R_s}\right) e^{r/R_s} \ .$$

Under the transformation (1.16), we have obtained a new metric (1.17), which is well-defined on $\mathcal{H}$. Once that the two-dimensional behaviour is known, the extension to the four-dimensional case becomes straightforward, actually leading to

$$d\tilde{s}^2 = -\frac{4R_s^3 e^{-r/R_s}}{r} \, dU dV + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \ . \tag{1.18}$$

This result shows that, thanks to some coordinate transformation, it is possible to map the Schwarzschild metric $\boldsymbol{g}$, given by (1.10), to a new metric $\widetilde{\boldsymbol{g}}$, given by (1.18), which is not afflicted by any singularity on the event horizon $\mathcal{H}$, being regular for $r \to R_s$. However, through the extension of Schwarzschild solution we have gained more than required. The extension $(\widetilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, which is called *Kruskal spacetime*, is a solution to the vacuum Einstein equation on $0 < r < +\infty$.

Before discussing the property of the Kruskal extension, we note that further improvements can be obtained from (1.18), by mapping the null Kruskal coordinates (U, V) to a new set (T, R)

$$T = \frac{1}{2}(U + V) \ ,$$
$$R = \frac{1}{2}(V - U) \ .$$

Since we will be interested in working with null geodesics, we do not discuss the details of this procedure, which has particular importance in showing, for instance, that the Kruskal solution mimics the asymptotic behaviour of the Minkowski spacetime. This property, is typically understood by means of a conformal mapping of (1.18) into a bounded region, which is usually pictured by the Penrose diagram 1.4 [15, Ch. 5].

We end this section by briefly making some considerations on the properties of the Kruskal extension, together with some useful definition.

1. In figure 1.2 every couple (U, V) represents a two-sphere. We will usually refer to the special case provided by the origin $(0, 0)$ as a *bifurcation sphere*.

2. We call *future* and *past horizon*, the hypersurfaces $\mathcal{H}^+$ and $\mathcal{H}^-$, which can be respectively obtained by taking the limit $U \to 0$ and $V \to 0$. We simply call *event horizon* the union $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$, which can be obtained by evaluating $r = R_s$, or equivalently, $UV = 0$.

3. The null coordinates (U, V) describe the light-cone structure of the Kruskal spacetime, which is centered on the bifurcation sphere. This fact comes naturally by observing that the radial null geodesics are all parallel to U or V. Indeed, we call *null outgoing* geodesics the V-directed null geodesics, while *null ingoing* geodesics those that are U-directed. See figure 1.3 for an example.

4. Regions II and III are respectively called *black hole* and *white hole*. Actually, the physical singularity $r = 0$ is mapped into the equation $UV = 1$.

5. The Schwarzschild solution (1.10) can be recovered as a restriction of the Kruskal spacetime to region I. Regions IV and III, can be physically interpreted in terms of the time reversal of I and II. For this reason, the Kruskal solution is said to describe an *eternal black hole*. However, this thesis is devoted to the analysis of those physical black holes which are a result of a complete stellar collapse. The presence of collapsing matter in the past is associated with a Kruskal diagram with the III and IV regions covered up [25].

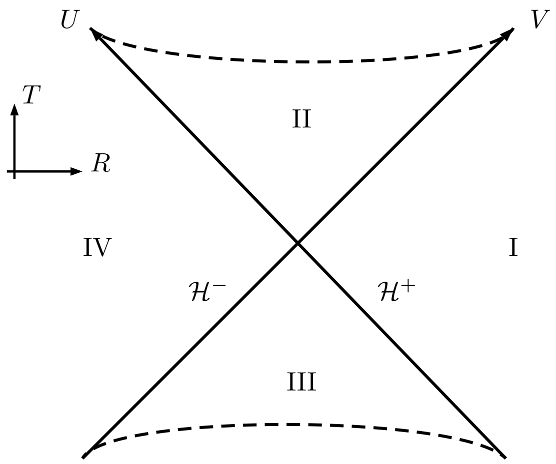


Figure 1.2: Kruskal diagram

1.4 The Raychaudhuri's equation

The purpose of our entire work is to give a fundamental description of black hole evaporation. Contrary to the classical case, where all the geodesics inside the event horizon remains trapped within its surface, the evaporation process is based on the possibility of a geodesic to actually violate this constraint, crossing the horizon and producing a flux of outgoing energy.

In this section we would like to develop the mathematical instruments necessary to understand the relation between the geometry of the background and the shape of a bunch of geodesics.

Let $(\mathcal{M}, \boldsymbol{g})$ be a spacetime, and O an open region in $\mathcal{M}$. A congruence in O is a family of curves such that through each point of O there passes one and only one curves of the family.

Strictly speaking a congruence represents a bundle whose elements never intersect each other within O, hence it is possible to associate a vector field $\boldsymbol{k}$, tangent in every point to the curves of the whole family. For our purposes we restrict our attention to *null geodesics congruence*, whose tangent vector field satisfies

$$\begin{aligned} g_{ab}k^a k^b &= 0 \ , \\ k^a \nabla_a k^b &= 0 \ . \end{aligned} \tag{1.19}$$

The role of general relativity is to link the geometry of a geodesic, i.e the word-line of a particle, with the notion of curvature of the spacetime. This also holds for a congruence of geodesics, whose shape can be completely described by the following tensor field [48, Ch. 9]

$$B_{ab} \doteq \nabla_b k_a \ . \tag{1.20}$$

The physical meaning of $\boldsymbol{B}$ comes up when considering an orthogonal infinitesimal displacement vector, which generates a vector field $\boldsymbol{\eta}$ such that $k^a \eta_a = 0$ and $\pounds_{\boldsymbol{k}} \boldsymbol{\eta} = 0$. Being

$$k^b \nabla_b \eta^a = B^a{}_b \eta^b \ ,$$

it follows that $B^a{}_b$ measure the failure of η^a to be parallel transported along the congruence, which depends on how the geodesics are locally deformed along the direction of $\boldsymbol{\eta}$.

Spacetime curvature may alter in several ways the shape of a family of geodesics. In order to extract all the geometrical information stored in $\boldsymbol{B}$, we need some preliminary consideration. From (1.20) and (1.19) it follows that

$$k^a B_{ab} = k^a B_{ba} = 0 \ , \tag{1.21}$$

showing that B_{ab} is completely fixed by the components which are transverse to the geodesics. Indeed, given a point p in $\mathcal{M}$, it may be natural to define

$$\boldsymbol{B}_p : V_p^\perp \otimes V_p^\perp \to \mathbb{R} \ , \tag{1.22}$$

with $V_p^\perp$ the vector space orthogonal to $\boldsymbol{k}$. However, contrary to the timelike case [48, Ch. 8], considering a null tangent requires additional mathematical care.

Indeed, if $\boldsymbol{\eta}$ is a vector of $V_p^\perp$ describing an infinitesimal deviation from the geodesic, then also $\boldsymbol{\eta} + c\boldsymbol{k}$ is an element of $V_p^\perp$, representing the same displacement. This fact brings additional degrees of freedom to $\boldsymbol{B}$, which are not physically significant. However, it also provides an equivalence relation such that

$$\eta_2^a \sim \eta_1^a \quad \text{if} \quad \eta_2^a = \eta_1^a + c k^a \ , \tag{1.23}$$

with $k_a k^a = 0$. Knowing (1.23), we can remove the redundant information from the definition (1.22), by restricting to the quotient space

$$\tilde{V}_p^\perp \doteq V_p^\perp / \sim \ ,$$

whose element are given by the equivalence classes

$$[\boldsymbol{\eta}] = \{\boldsymbol{\eta} + c\boldsymbol{k} \mid k_a k^a = 0, \ \boldsymbol{\eta} \in V_p^\perp\} \ . \tag{1.24}$$

This leads to the following "reduced" tensor

$$\hat{\boldsymbol{B}}_p : \tilde{V}_p^\perp \otimes \tilde{V}_p^\perp \to \mathbb{R} \ ,$$

whose components can be obtained from (1.22), by means of a projector over the quotient space

$$h : V_p^{\perp} \to \tilde{V}_p^{\perp} \ , \tag{1.25}$$

such that

$$\hat{B}_{ab} = h_a{}^c h_b{}^d B_{cd} \ . \tag{1.26}$$

We represent the equivalent classes (1.24) and the projector (1.25), by choosing a suitable auxiliary vector field $\boldsymbol{l}$ [43, Ch. 2], [9, App. F], which satisfies $k^a l_a = -1$, thus leading to

$$h_{ab} = g_{ab} + k_a l_b + l_a k_b \ , \tag{1.27}$$

which by definition satisfies

$$k^a h_{ab} = k^a h_{ba} = 0 \ , \tag{1.28}$$

Indeed, by substitution of (1.27) into (1.20), we get

$$\hat{B}_{ab} = B_{ab} + k_a l^c B_{cb} + l^c B_{ac} k_b + k_a k_b l^c l^d B_{cd} \ .$$

Physically, we should note that, by inverting (1.27) it follows that $g_{ab} = +h_{ab} - k_a l_b - l_a k_b$. Which actually represents the decomposition of the metric $\boldsymbol{g}$ into its transverse component $\boldsymbol{h}$ and its longitudinal part $\boldsymbol{k}\otimes\boldsymbol{l}$, with respect to the geodesics flow [43, Ch. 2].

Finally, we use the previous results to classify the geometrical information stored in $\hat{\boldsymbol{B}}$, by actually use the transverse metric to get the following decomposition

$$\hat{B}_{ab} = \frac{1}{2}\theta h_{ab} + \hat{\sigma}_{ab} + \hat{\omega}_{ab} \ , \tag{1.29}$$

We call *congruence parameters* the different contributions to this decomposition, which are given by

$$\begin{aligned}
\theta &= h^{ab}\hat{B}_{ab} \ , \\
\hat{\omega}_{ab} &= \hat{B}_{[ab]} \ , \\
\hat{\sigma}_{ab} &= \hat{B}_{(ab)} - \frac{1}{2}\theta h_{ab} \ ,
\end{aligned} \tag{1.30}$$

with the symmetrization and anti-symmetrization of $\hat{B}_{ab}$ respectively given by

$$\begin{aligned}
\hat{B}_{(ab)} &\doteq \tfrac{1}{2}(\hat{B}_{ab} + \hat{B}_{ba}) \ , \\
\hat{B}_{[ab]} &\doteq \tfrac{1}{2}(\hat{B}_{ab} - \hat{B}_{ba}) \ .
\end{aligned}$$

The congruence parameters θ, $\hat{\sigma}_{ab}$ and $\hat{\omega}_{ab}$ have different physical interpretations. Let us consider, for instance, a three-dimensional flat spacetime with Cartesian coordinates $\{x, y, z\}$. Let $\boldsymbol{C}$ be a two-dimensional congruence, made by a single circumference on the xy plane. The *shear tensor* $\hat{\sigma}_{ab}$ describes how $\boldsymbol{C}$ can be deformed to an ellipse with arbitrary orientation on xy. On the other hand, the effect of the *torsion tensor* $\hat{\omega}_{ab}$ is associated to that of a rotation of $\boldsymbol{C}$ with respect to the z-axis. Finally, the *expansion parameter* θ encodes the variation of area of the circle enclosed by $\boldsymbol{C}$, as a consequence of an expansion or compression of its perimeter.

This geometrical interpretation can also be extended to Lorentzian manifolds, where the deformation, rotation and expansion is not constant, being generated by spacetime curvature. In the following section, we will give further details regarding the physical interpretation of the expansion parameter of a null congruence, in terms of the rate of variation of the area of a black hole [43, Ch. 2].

The relation between the geodesics parameters (1.30) and the geometrical property of the spacetime can be shown as follows. Let us consider a geodesics null congruence with affine parameter λ. Taking the derivative of $\boldsymbol{B}$ (1.20) along a geodesic, it follows that [48]

$$\frac{dB_{ab}}{d\lambda} = k^c \nabla_c B_{ab} = -B^c{}_b B_{ac} + R_{cbad} k^c k^d \ .$$

By a two-fold multiplication of both members for $\boldsymbol{h}$, we get the "hatted" equation

$$k^c \nabla_c \hat{B}_{ab} = -\hat{B}^c{}_b \hat{B}_{ac} + \widehat{R_{cbad} k^c} k^d \ . \tag{1.31}$$

By substitution of the decomposition of $\hat{\boldsymbol{B}}$ (1.29), we get the *Raychaudhuri's equation* for a congruence of null geodesics

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{cd} k^c k^d \ . \tag{1.32}$$

Using the alternative form of the Einstein equation (1.5) and recalling that for a congruence of null geodesics $g_{ab} k^a k^b = 0$, we can account for the contribution of matter as

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - 8\pi T_{cd} k^c k^d \ ,$$

with $G = 1$. Taking respectively the symmetric and anti-symmetric part of (1.31), we get

$$\begin{aligned} \frac{d\hat{\sigma}_{ab}}{d\lambda} &= -\theta\hat{\sigma}_{ab} + \widehat{C_{cbad} k^c} k^d \ , \\ \frac{d\hat{\omega}_{ab}}{d\lambda} &= -\theta\hat{\omega}_{ab} \ , \end{aligned} \tag{1.33}$$

with $\boldsymbol{C}$ being the *Weyl tensor*, which in four-dimensional manifolds takes form [48, Ch. 8]

$$C_{abcd} = R_{abcd} - \left(g_{a[c} R_{d]b} + g_{b[c} R_{d]a} \right) - \frac{1}{3} R g_{a[c} g_{d]b} \ . \tag{1.34}$$

1.4.1 The physical meaning of the expansion parameter

In this section we would like to highlight the physical interpretation of the expansion parameter θ, in the case of a congruence of null geodesics [43, Ch. 2].

Let us consider a congruence of null geodesics, with tangent vector field $\boldsymbol{k}$ and affine parameter λ. Let $\boldsymbol{\gamma}$ be a geodesic of the congruence, and p a particular point of $\boldsymbol{\gamma}$ at which $\lambda = \lambda_p$. Moreover, we consider the family of null curves, parametrized by μ, to which the auxiliary vector field $\boldsymbol{l}$ of (1.27) is tangent. We denote $\boldsymbol{\alpha}$ the particular auxiliary curve, which passes through p with $\mu = \mu_p$.

The *cross section* S_p is a neighbourhood of p, i.e a set of points $q \in \mathcal{M}$ sufficiently close to p, such that, for any q there passes another geodesic of the congruence $\boldsymbol{\gamma}_q$

and another auxiliary curve $\boldsymbol{\alpha}_q$ which respectively gives $\lambda_q = \lambda_p$ and $\mu_q = \mu_p$. Intuitively, given $p \in \mathcal{M}$, S_p is obtained as the collection of those points which are spanned by the congruence (auxiliary family) with constant value of the affine parameter $\lambda = \lambda_p$ ($\mu = \mu_p$).

A cross section describes a two-dimensional surface, which depends only upon a set of two-dimensional coordinates Ω^A_p, with $A = 2, 3$. If the spacetime is spherically symmetric and if the chosen congruence is made of radial geodesics, the cross section S_p is naturally labelled by the angular coordinates of (1.10)

$$\Omega^2 = \theta \ , \quad \Omega^3 = \varphi \ .$$

Since for each point of S_p there passes a geodesic of the congruence, we can use Ω^A_p to label the deviation from $\boldsymbol{\gamma}$ to the other elements of the family, which intersect S_p at λ_p. Assuming that the label of each geodesics is preserved when moving away from S_p, we can get a set of coordinate Ω^A for any cross-section $S(\lambda)$.

This discussion suggests that we can label the four-dimensional coordinates on the manifold $\mathcal{M}$ as $x^a(\lambda, \mu, \Omega^2, \Omega^3)$[2]. Let us consider the following collection of vectors labelled the two-dimensional index A

$$e^a_A \doteq \left.\frac{\partial x^a}{\partial \Omega^A}\right|_{\lambda,\mu=\text{cost.}} ,$$

which are tangent to $S(\lambda)$. We can define the following two-dimensional matrix

$$\mathscr{G}_{AB} = g_{ab} e^a_A e^b_B \ , \tag{1.35}$$

which acts as a (angular) metric on the cross-section. Because $\boldsymbol{\gamma}$ is orthogonal to its cross section, we get that

$$k_a e^a_A = 0 \ .$$

By substitution of (1.27) in (1.35), we immediately get

$$\mathscr{G}_{AB} = h_{ab} e^a_A e^b_B \ .$$

Hence, by denoting the $\mathscr{G}^{AB}$ the inverse of the angular metric (1.35), we get

$$h^{ab} = \mathscr{G}^{AB} e^a_A e^b_B \ ,$$

with $\boldsymbol{h}$ as in (1.27), it follows that

$$\frac{d\mathscr{G}_{AB}}{d\lambda} = 2B_{(ab)} e^a_A e^b_B \ . \tag{1.36}$$

Defining the cross sectional infinitesimal area as $dA = \det[\mathscr{G}]^{1/2} d^2\Omega$, we multiply (1.36) by the inverse angular metric $\mathscr{G}^{AB}$. Finally, using definition (1.30), we get

$$\theta = \frac{1}{A}\frac{dA}{d\lambda} \tag{1.37}$$

Hence, the expansion parameter measures the rate of change of the congruence's cross-sectional area, which in our case will represent the area of the black hole event horizon.

[2] Assuming that the auxiliary parameter μ has been chosen to be constant along the geodesics and such that both $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ intersect S_p orthogonally.

1.4.2 Geodesics congruence on Kruskal background

In this section we briefly discuss the properties of a congruence of geodesics restricted to event horizon of a static spherically-symmetric black hole. Our purpose is to highlight the usefulness of the Raychaudhuri's equation, as far as concerning the analysis of the evaporation of a black hole.

We are going to work with a Kruskal background (1.18), which we recall being

$$ds^2 = -\frac{32M^3}{r}e^{-r/2M}dUdV + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2 \ , \tag{1.38}$$

with $G = 1$ and $R_s = 2M$. We recall that $\mathcal{H}^+$ and $\mathcal{H}^-$ denote the future and past horizon, which can be respectively obtained by taking the limit $U \to 0$ and $V \to 0$.

We consider a congruence of radial outgoing null geodesics with tangent vector field

$$\boldsymbol{k} = f(U,V)\partial_V \ . \tag{1.39}$$

To compute the Raychaudhuri's equation, we are interested in determining the

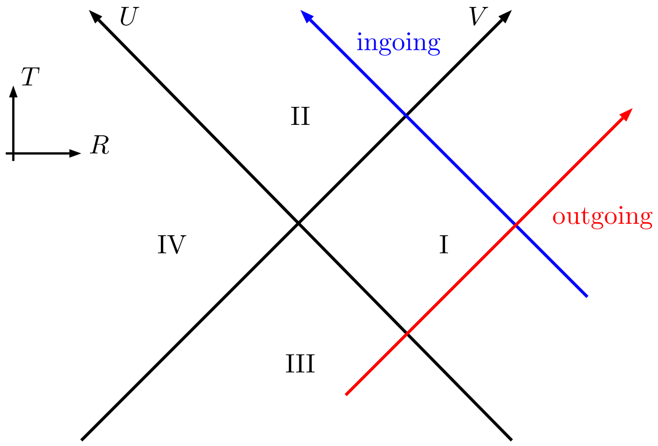


Figure 1.3: Ingoing and outgoing radial null geodesics

restriction of $\boldsymbol{B}$ to $\mathcal{H}^+$. A priori, for the congruence (1.39), there are only two non-vanishing component of $\boldsymbol{B}|_{\mathcal{H}^+}$

$$\begin{aligned} B_{UV}|_{\mathcal{H}^+} &= -\frac{8M^2}{e}\partial_V f(0,V) \ , \\ B_{UU}|_{\mathcal{H}^+} &= -\frac{8M^2}{e}\partial_U f(0,V) \ , \end{aligned} \tag{1.40}$$

with e the Euler's number, obtained by the exponential function in (1.38), under the limit $r \to 2M$.[3]

The vector field $\boldsymbol{k}$ satisfies the geodesic equation $k^a\nabla_a k^b = 0$, which, on the future horizon, gives

$$f(0,V)\partial_V f(0,V) = 0 \ .$$

[3]This observation is enough to conclude that the congruence parameters associated to (1.39) vanish on the event horizon. However, we prefer going a little further, exploiting (1.40) to get an explicit expression of $\boldsymbol{B}|_{\mathcal{H}^+}$.

Asking $f(0,V)\neq 0$, we get that

$$\partial_V f(0,V)=0\ , \tag{1.41}$$

hence $B_{UV}|_{\mathcal{H}^+}=0$. Now we can proceed with the explicit computation of the $B_{UU}|_{\mathcal{H}^+}$ component.

The geodesic equation requires a proper choice of initial data with respect to the affine parameter λ, setting up the following initial value problem

$$\frac{d\boldsymbol{k}}{d\lambda}=0\ ,\quad (\boldsymbol{k}\cdot\boldsymbol{\partial_U})|_{\mathcal{H}^-}=\alpha\ .$$

Here, we have chosen $\mathcal{H}^-$ as a surface with constant λ. Exploiting the initial condition we get that $fg_{UV}|_{\mathcal{H}^-}=\alpha$, ensuring

$$\begin{aligned} f(U,0)&=-\frac{\alpha e}{8M^2}\ ,\\ \partial_U f(U,0)&=0\ . \end{aligned} \tag{1.42}$$

Evaluating this last result on the bifurcation sphere together with the condition (1.41), gives

$$f(0,V)=-\frac{\alpha e}{8M^2}\ . \tag{1.43}$$

On the other hand, we consider explicitly the non-trivial component of the geodesic equation

$$f(U,V)\left(\partial_V f(U,V)+\frac{(r+2M)2M}{r^2}Uf(U,V)e^{-r/2M}\right)=0\ .$$

We derive both members with respect to U, then taking the limit to $\mathcal{H}^+$. Using (1.41) and being $f(0,V)\neq 0$, we get

$$\frac{2}{e}f(0,V)+\partial_V\partial_U f(0,V)=0\ .$$

This equation can be integrated along $\mathcal{H}^+$. Exploiting (1.42) on the bifurcation sphere to neglect the constant term, we obtain

$$\partial_U f(0,V)=\frac{\alpha V}{4M^2}\ ,$$

which finally fixes $B_{UU}|_{\mathcal{H}^+}$.

We can resume the results of this section stating that for (1.39)

$$\boldsymbol{B}|_{\mathcal{H}^+}=\frac{-2\alpha V}{e}dU\otimes dV\ ,$$

which automatically leads to

$$\widehat{\boldsymbol{B}}|_{\mathcal{H}^+}=0\ .$$

Recalling the definitions of the congruence parameters (1.30), we have finally proven that for radial outgoing null congruence on a Kruskal background

$$\theta|_{\mathcal{H}^+}=0,\quad \hat{\sigma}_{\mu\nu}|_{\mathcal{H}^+}=0\ ,\quad \hat{\omega}_{\mu\nu}|_{\mathcal{H}^+}=0\ .$$

This results, which may appear trivial, confirms the stability of $\mathcal{H}$ of a static spherically symmetric black hole, which cannot spontaneously evaporate from a classical point of view. It is quite natural to ask what happens when we account for the presence of quantum gravitational radiation.

1.5 Motivating the perturbative approach

In this section we consider the example of a black hole, subject to a free gravitational perturbation. We would like to adopt a perturbative point of view, exploiting all the information that can be studied at linear order.

We denote the complete spacetime, accounting for both the geometry of the black hole and the contribution of the gravitational wave, as $(\mathcal{M}, \tilde{\boldsymbol{g}})$. Hence, we consider the following linearization of the complete metric

$$\tilde{g}_{ab} = g_{ab} + \varepsilon\gamma_{ab} + \mathcal{O}(\epsilon^2) \ , \tag{1.44}$$

with $\boldsymbol{g}$ being the Kruskal metric. We consider the effect of the gravitational perturbation on the spherically-symmetric background, by using prescription (1.44) to give the expansion in terms of powers of $\boldsymbol{\gamma}$ of the congruence parameters, such that

$$\theta = \sum_{i=0}^{+\infty} \theta^{(i)}\varepsilon^i \ , \quad \hat{\sigma}_{ab} = \sum_{i=0}^{+\infty} \hat{\sigma}_{ab}^{(i)}\varepsilon^i \ , \quad \hat{\omega}_{ab} = \sum_{i=0}^{+\infty} \hat{\omega}_{ab}^{(i)}\varepsilon^i \ .$$

as well as for the Ricci tensor and for the geodesics tangent vector field

$$k^a = \sum_{i=0}^{+\infty} k^a_{(i)}\varepsilon^i \ , \quad R_{ab} = \sum_{i=0}^{+\infty} R_{ab}^{(i)}\varepsilon^i \ .$$

We assume that, on the background spacetime, $\boldsymbol{k^{(0)}}$ describes a vector field tangent to a radial null outgoing geodesics congruence. According to the discussion made in the previous section, the restriction of such a congruence to the horizon of a Kruskal background, gives

$$\theta^{(0)}|_{\mathcal{H}^+} = 0, \quad \hat{\sigma}^{(0)}_{\mu\nu}|_{\mathcal{H}^+} = 0 \ , \quad \hat{\omega}^{(0)}_{\mu\nu}|_{\mathcal{H}^+} = 0 \ , \tag{1.45}$$

moreover, since we are considering a free gravitational perturbation propagating along a vacuum spacetime, we get

$$R^{(0)}_{\mu\nu}|_{\mathcal{H}^+} = 0 \ , \quad R^{(1)}_{\mu\nu}|_{\mathcal{H}^+} = 0 \ .$$

By substitution into the RHS of the Raychaudhuri's equation (1.32) and of (1.33), which are quadratic in the congruence parameters, we get

$$\left.\frac{d\theta}{d\lambda}\right|^{(1)}_{\mathcal{H}^+} = 0 \ , \quad \left.\frac{d\hat{\omega}_{ab}}{d\lambda}\right|^{(1)}_{\mathcal{H}^+} = 0 \ .$$

Assuming that the horizon is stable in the past, i.e that gravitational perturbations come from the past-null infinity, we can stress that

$$\theta^{(1)}|_{\mathcal{H}^+} = 0 \ , \quad \hat{\omega}^{(1)}_{ab}|_{\mathcal{H}^+} = 0 \ .$$

The previous statements allow us to conclude that

$$\begin{aligned} \theta|_{\mathcal{H}^+} &= \varepsilon^2\theta^{(2)} + \mathcal{O}(\varepsilon^3) \ , \\ \hat{\omega}_{ab}|_{\mathcal{H}^+} &= \varepsilon^2\hat{\omega}^{(2)}_{ab} + \mathcal{O}(\varepsilon^3) \ , \end{aligned} \tag{1.46}$$

which gives

$$\theta^2 = \mathcal{O}(\varepsilon^4) \ , \quad \hat{\omega}_{ab}\hat{\omega}^{ab} = \mathcal{O}(\varepsilon^4) \ .$$

Hence, the second order contributions to the Raychaudhuri's equation (1.32) come only from the first order correction to the shear tensor and from the second order correction to the Ricci tensor, thus

$$\left.\frac{d\theta}{d\lambda}\right|_{\mathcal{H}^+} = -\varepsilon^2 \hat{\sigma}^{(1)}_{ab}\hat{\sigma}^{ab}_{(1)} - \varepsilon^2 R^{(2)}_{cd} k^c k^d + \mathcal{O}(\varepsilon^3) \ . \tag{1.47}$$

From (1.47) we observe that linear perturbations of the metric are enough to study second order effects connected to Hawking radiation on the event horizon. Indeed, (1.47) shows that γ introduces some potential instability on $\mathcal{H}$, which may lead to a non-vanishing rate of energy crossing the horizon, being triggered by the emission of Hawking radiation.

The following chapters will be devoted to the description of gravitational perturbation in terms of a quantum field theory of linearized gravity. The algebraic approach will be developed, providing powerful techniques which will be employed in the computation of backreaction involved in (1.47).

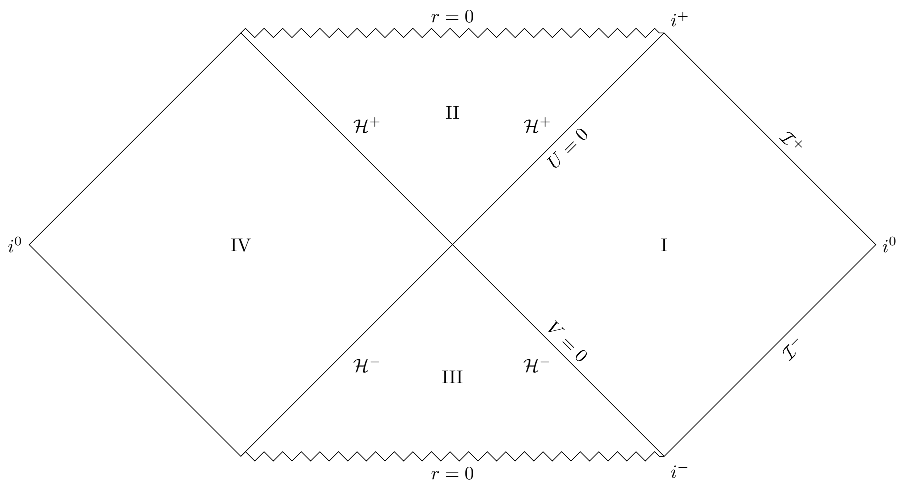


Figure 1.4: A Penrose diagram of the Kruskal spacetime

Part II

Linearized gravity and algebraic quantization

Chapter 2

Classical theory

2.1 Introduction

In this part of the thesis we shall discuss the quantization of linearized gravity from the point of view of algebraic quantum field theory on curved spacetimes.

This first chapter will be devoted to the construction of the classical theory, starting from the expansion of the linear perturbation field with respect to a fixed background spacetime. Then, the linearization of the Einstein equation will be reviewed. Moreover, the gauge-invariance carried by linearized diffeomorphisms will be exploited, thus obtaining a set of normally hyperbolic equations of motion, which drives the dynamics of the perturbation field on the geometrical background. A suitable choice of the initial data will guarantee the existence and uniqueness of solutions to the field equation, and thus of the causal propagator, which will play a pivotal role in the second chapter, where the quantization of the theory will be achieved.

2.2 Linearization of the Einstein tensor

Let us consider a one-parameter family of solutions of the Einstein field equations $\varepsilon \to \tilde{\boldsymbol{g}}(\varepsilon)$, such that $\tilde{\boldsymbol{g}}(0) \doteq \boldsymbol{g}$ and $\tilde{\boldsymbol{g}}'(0) \doteq \boldsymbol{\gamma}$. To compute the linearized Einstein equation, we first study the one-parameter family of Einstein tensor $\varepsilon \to \widetilde{\boldsymbol{G}}(\tilde{\boldsymbol{g}})(\varepsilon)$, induced by the expansion of the metric, such that $\widetilde{\boldsymbol{G}}(\tilde{\boldsymbol{g}})(0) \doteq \boldsymbol{G}(g)$ and $\widetilde{\boldsymbol{G}}(\tilde{\boldsymbol{g}})'(0) \doteq \boldsymbol{L}(\boldsymbol{g};\gamma)$. Hence, we make the following linearization

$$\tilde{g}_{ab} = g_{ab} + \varepsilon\gamma_{ab} + \mathcal{O}(\varepsilon^2) \ . \tag{2.1}$$

At this level, we are not considering any particular choice of $\boldsymbol{g}$, that describes the classical curved background on which the gravitational radiation $\boldsymbol{\gamma}$ will propagate.

Under (2.1), two possible choice of spacetimes arise. The *background spacetime* $(\mathcal{M}, \boldsymbol{g})$, which at the moment remains fixed, and the *complete spacetime* $(\widetilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, which accounts for both the contributions coming from the background geometry and the gravitational radiation. From now on, we will denote any object related to the complete or background spacetime respectively with or without a tilde, referring to it as a complete or background quantity, e.g $\widetilde{\boldsymbol{G}}$ as the *complete Einstein tensor*.

We adopt the conventions of [48], using the background metric g_{ab} to raise and lower all the indices, without worrying about the presence of hidden γ_{ab} terms.

Hence, the raised complete metric reads

$$\tilde{g}^{ab} = g^{ab} - \varepsilon\gamma^{ab} + \mathcal{O}(\varepsilon^2) \ .$$

2.2.1 Complete and background covariant derivatives

We follow the idea of [48, Ch. 3], allowing us to compute the *complete covariant derivative* $\widetilde{\nabla}_\mu$ with respect to the background one ∇_μ. Indeed, given a vector $\boldsymbol{V}$, the difference between two covariant derivatives defines a tensor of type $(1,2)$, such that

$$\widetilde{\nabla}_\mu V^\lambda = \nabla_\mu V^\lambda + \widetilde{C}^\lambda{}_{\mu\nu} V^\nu \ . \tag{2.2}$$

From now on, we assume that both the spacetime and the background are torsionless, which guarantees that $\widetilde{C}^\lambda{}_{\mu\nu}$ is symmetric under the exchange of the lower indices.

The action of the complete covariant derivative on a (k,l) tensor is

$$\begin{aligned}\widetilde{\nabla}_\mu T^{\lambda_1\dots\lambda_k}{}_{\nu_1\dots\nu_l} =& \nabla_\mu T^{\lambda_1\dots\lambda_k}{}_{\nu_1\dots\nu_l} + \sum_i \widetilde{C}^{\lambda_i}{}_{\mu\rho} T^{\lambda_1\dots\rho\dots\lambda_k}{}_{\nu_1\dots\nu_l} + \\ &- \sum_i \widetilde{C}^{\rho}{}_{\mu\nu_i} T^{\lambda_1\dots\lambda_k}{}_{\nu_1\dots\rho\dots\nu_l} \ .\end{aligned} \tag{2.3}$$

We take the case of the metric, where we have that

$$\widetilde{\nabla}_\mu \tilde{g}_{\nu\rho} = \nabla_\mu \tilde{g}_{\nu\rho} - \widetilde{C}^\lambda{}_{\mu\nu}\tilde{g}_{\lambda\rho} - \widetilde{C}^\lambda{}_{\mu\rho}\tilde{g}_{\nu\lambda} \ .$$

Ensuring the validity of the Levi-Civita theorem, we require the metric-compatibility condition on the complete spacetime, which reads $\widetilde{\nabla}_\mu \tilde{g}_{\nu\rho} = 0$. By index substitution, we get three independent equations

$$\nabla_\mu \tilde{g}_{\nu\rho} = \widetilde{C}^\lambda{}_{\mu\nu}\tilde{g}_{\lambda\rho} + \widetilde{C}^\lambda{}_{\mu\rho}\tilde{g}_{\nu\lambda} \tag{2.4}$$

$$\nabla_\rho \tilde{g}_{\mu\nu} = \widetilde{C}^\lambda{}_{\rho\mu}\tilde{g}_{\lambda\nu} + \widetilde{C}^\lambda{}_{\rho\nu}\tilde{g}_{\mu\lambda} \tag{2.5}$$

$$\nabla_\nu \tilde{g}_{\rho\mu} = \widetilde{C}^\lambda{}_{\nu\rho}\tilde{g}_{\lambda\mu} + \widetilde{C}^\lambda{}_{\nu\mu}\tilde{g}_{\rho\lambda} \ . \tag{2.6}$$

Adding (2.4) to (2.5), subtracting (2.6) and solving, we get[1]

$$\widetilde{C}^\lambda{}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\lambda\alpha}(\nabla_\mu \tilde{g}_{\alpha\nu} + \nabla_\nu \tilde{g}_{\mu\alpha} - \nabla_\alpha \tilde{g}_{\mu\nu}) \ .$$

We substitute (2.1), requiring the background metric-compatibility condition $\nabla_\mu g_{\nu\rho} = 0$. Then, we can write $\widetilde{C}^\lambda{}_{\mu\nu}$ in terms of the perturbation field

$$\widetilde{C}^\lambda{}_{\mu\nu} = \frac{\varepsilon}{2} g^{\lambda\alpha}(\nabla_\mu \gamma_{\alpha\nu} + \nabla_\nu \gamma_{\mu\alpha} - \nabla_\alpha \gamma_{\mu\nu}) + \mathcal{O}(\varepsilon^2) \ . \tag{2.7}$$

To clarify future dependences from the infinitesimal parameter ε, we give the following definition, by simply removing the tilde prescription and the ε factor from $\widetilde{C}^\lambda{}_{\mu\nu}$

$$C^\lambda{}_{\mu\nu} \doteq \frac{1}{2} g^{\lambda\alpha}(\nabla_\mu \gamma_{\alpha\nu} + \nabla_\nu \gamma_{\mu\alpha} - \nabla_\alpha \gamma_{\mu\nu}) \ . \tag{2.8}$$

With a little of effort we have obtained a way to link the complete covariant derivative to the geometry of the background spacetime which, in some sense, involves just the Christoffel symbols associated to the perturbation $\boldsymbol{\gamma}$.

[1] Actually there isn't any preferred choice of which on the three equations need to be subtracted from the sum of other ones

Properties We end this section by discussing two properties of the complete covariant derivative, which will simplify the computation of the linearized Riemann tensor.

1. We insert (2.7) in (2.3). Using[2] that $\varepsilon\widetilde{\nabla}_\mu g^{\lambda\rho} = \mathcal{O}(\varepsilon^2)$, we get
$$\widetilde{\nabla}_\mu \widetilde{C}^\lambda{}_{\nu\rho} = \varepsilon \nabla_\mu C^\lambda{}_{\nu\rho} + \mathcal{O}(\varepsilon^2) \ . \tag{2.9}$$

2. From (2.3) the action of the covariant derivative on a one-form $\boldsymbol{\omega}$ is given by
$$\widetilde{\nabla}_\mu \omega_\nu = \nabla_\mu \omega_\nu - \widetilde{C}^\lambda{}_{\mu\nu}\omega_\lambda,$$
so it follows that
$$\widetilde{C}^\lambda{}_{\nu\rho}\widetilde{\nabla}_\mu \omega_\lambda = \varepsilon C^\lambda{}_{\nu\rho}\nabla_\mu\omega_\lambda + \mathcal{O}(\varepsilon^2) \tag{2.10}$$

2.2.2 The linearization of the geometrical quantities

We are now ready to compute the linearization of the *complete Riemann tensor*. Starting from its definition
$$\widetilde{R}_{\mu\nu\rho}{}^\lambda \omega_\lambda = 2\widetilde{\nabla}_{[\mu}\widetilde{\nabla}_{\nu]}\omega_\rho \ ,$$
we expand the covariant derivative twice. Having observed that $\nabla_\mu\omega_\nu$ is a $(0,2)$ tensor and thanks to some simplification connected to property (2.9) and (2.10), we get that
$$\widetilde{R}_{\mu\nu\rho}{}^\lambda = R_{\mu\nu\rho}{}^\lambda - 2\varepsilon\nabla_{[\mu}C^\lambda{}_{\nu]\rho} + \mathcal{O}(\varepsilon^2) \ ,$$
where $R_{\mu\nu\rho}{}^\lambda := 2\nabla_{[\mu}\nabla_{\nu]}\omega_\rho$ is the background Riemann tensor.

We recall the definition of the *complete Ricci tensor* $\widetilde{R}_{\mu\nu} := \widetilde{R}_{\mu\lambda\nu}{}^\lambda$. By substitution of $C^\lambda{}_{\mu\nu}$
$$\widetilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{\varepsilon}{2}\left(\nabla_\mu\nabla_\nu\gamma + \Box\gamma_{\mu\nu} - 2g^{\alpha\beta}\nabla_\alpha\nabla_{(\mu}\gamma_{\nu)\beta}\right) + \mathcal{O}(\varepsilon^2) \ ,$$
where $tr(\gamma) \doteq g^{\mu\nu}\gamma_{\mu\nu} = \gamma^\mu{}_\mu$ and $\Box := g^{\alpha\beta}\nabla_\alpha\nabla_\beta$. We trace by $\tilde{g}^{\mu\nu}$, getting the *complete Ricci scalar*
$$\widetilde{R} = R - \varepsilon(\Box\gamma^\alpha{}_\alpha - \nabla_\alpha\nabla_\beta\gamma^{\alpha\beta}) + \mathcal{O}(\varepsilon^2)$$
Again, both can be viewed as first order corrections to $R_{\mu\nu} := R_{\mu\lambda\nu}{}^\lambda$ and $R := g^{\mu\nu}R_{\mu\nu}$, respectively the background Ricci tensor and curvature.

The complete Einstein tensor is defined as
$$\widetilde{G}_{\mu\nu} \doteq \widetilde{R}_{\mu\nu} - \frac{1}{2}\widetilde{R}\tilde{g}_{\mu\nu} \ .$$
By substitution of our previous results, we get the expansion $\widetilde{G}_{\mu\nu} = G_{\mu\nu} + \varepsilon L_{\mu\nu} + \mathcal{O}(\varepsilon^2)$ with
$$L_{\mu\nu} = \nabla_\alpha\nabla_{(\mu}\gamma_{\nu)}{}^\alpha - \frac{1}{2}\nabla_\mu\nabla_\nu\gamma - \frac{1}{2}\Box\gamma_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box\gamma - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta\gamma^{\alpha\beta} \ . \tag{2.11}$$

[2] Again this comes directly from the definition of the covariant derivative and the background metric-compatibility condition, together with the observation that $C^\lambda{}_{\mu\nu}$ is pure first-order term

We make the following redefinition

$$\bar{\gamma}_{\mu\nu} \doteq \gamma_{\mu\nu} - \frac{1}{2}\operatorname{tr}(\gamma) g_{\mu\nu} \ , \tag{2.12}$$

which put (2.11) in the following form

$$L_{\mu\nu} = \nabla_\alpha \nabla_{(\mu} \bar{\gamma}_{\nu)}{}^\alpha - \frac{1}{2}\Box\bar{\gamma}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \nabla_\beta \bar{\gamma}^{\alpha\beta} \ . \tag{2.13}$$

We have completed the linearization of the Einstein tensor. The expression (2.11) justifies our initial prescription $\boldsymbol{L}(g;\gamma)$ since (2.13) acts like a tensor partial differential operator on $\boldsymbol{\gamma}$, while carrying a parametric dependence from $\boldsymbol{g}$, both explicitly and through the background covariant derivative.

Remark We point out that the expression (2.11), which holds for any choice of the background $\boldsymbol{g}$, can be viewed as the curved extension of the Einstein tensor linearized around the Minkowski background $\boldsymbol{\eta}$ [48, Ch. 4]. Indeed, $\widetilde{G}_{\mu\nu}$ can be obtained from the flat case by the formal substitution

$$\partial \to \nabla \ , \quad \boldsymbol{\eta} \to \boldsymbol{g} \ .$$

2.3 Equations of motion

The Einstein tensor has been linearized and split as a sum of two contributions: the background Einstein tensor $\boldsymbol{G}$ and the linearized tensor $\boldsymbol{L}$, which drives the dynamics of $\boldsymbol{\gamma}$.

We have all the necessary tools to write down the linearized vacuum Einstein equation. Starting from the complete vacuum Einstein equation

$$\boldsymbol{G}(\boldsymbol{g}) + \varepsilon \boldsymbol{L}(\boldsymbol{g};\boldsymbol{\gamma}) = 0 \ ,$$

and imposing (1.6) on the background, we can isolate the first order term, which reads

$$L_{ab}(\boldsymbol{g};\boldsymbol{\gamma}) = 0 \ . \tag{2.14}$$

As stressed before, we can interpret the action of $\boldsymbol{L}$ on $\boldsymbol{\gamma}$ in terms of a tensor differential operator, which generates the field equations of a gravitational perturbation on a curved background spacetime. This will be our starting point in the description of the classical theory of linearized gravity.

As we shall discuss in the following sections, the existence and uniqueness of the advanced and retarded propagators is ensured if the field equations are normally hyperbolic [3, 18, 22, 4]. However, this is not the case of $\boldsymbol{L}$ which does not appear in (2.11) in a normally hyperbolic form. Nevertheless, all is not lost. We exploit the gauge freedom of our theory to put $\boldsymbol{L}$ in the desired fashion.

2.3.1 Diffeomorphism and gauge invariance

Let us briefly discuss some aspect of gauge freedom for linearized gravity. Equation (2.14) has been obtained following a geometrical approach. However, it can be also derived by means of the action principle.

Let us consider the Einstein-Hilbert action [48, 9], by neglecting the cosmological constant Λ and any boundary term [48, App. E]

$$S_{EH} = \frac{1}{32\pi G}\int R\sqrt{-g}d^4x \ . \tag{2.15}$$

Expanding to second order, we get the action for linearized gravity on a curved background

$$\begin{aligned} S_{lg} = \frac{1}{32\pi G}\int [&\tfrac{1}{2}\gamma^{ab}\Box\gamma_{ab} - \tfrac{1}{4}\gamma^a{}_a\Box\gamma^b{}_b + (\nabla^a\gamma_{ab} - \tfrac{1}{2}\nabla_b\gamma^a{}_a)^2 + \gamma^{ab}R_{acbd}\gamma^{cd} + \\ &+ \gamma^a{}_b R^{bc}\gamma_{ac} - \gamma^c{}_c\gamma^{ab}R_{ab} - \tfrac{1}{2}R\gamma_{ab}\gamma^{ab} + \tfrac{1}{4}R(\gamma^a{}_a)^2]\sqrt{-g}d^4x \ . \end{aligned} \tag{2.16}$$

which reproduces the free field equations given by (2.11) [1, 18].

Let us briefly review the idea behind the covariance of general relativity and its consequences for gravitational perturbations. We consider a *diffeomorphism*

$$\phi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}} \ ,$$

namely an invertible map such that both the map and its inverse are smooth. Algebraically, covariance requires ϕ to be an automorphisms of the space of solution of the Einstein equation. Namely, if $(\widetilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is a solution of (1.4), then also $(\widetilde{\mathcal{N}}, \phi^*\tilde{\boldsymbol{g}})$ solves (1.4), with ϕ^* being the pullback generated by ϕ [48, App. C].

The request for covariance has a straight physical consequence: if two spacetimes are related by a diffeomorphism, then they are physically indistinguishable. Thus, the mathematical idea behind ϕ founds an interpretation in terms on the equivalence of the classical measures obtained by different observers of the same spacetime.

From an infinitesimal point of view, the action of a diffeomorphism ϕ can be studied in terms of a vector field $\boldsymbol{w}$, which generates the transformation [48, Ch. 2]. Let us consider a one-parameter group of diffeomorphisms

$$\phi_t : \mathbb{R} \times \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}} \ ,$$

such that $\phi_t \circ \phi_s = \phi_{t+s}$. Indeed, fixed $t \in \mathbb{R}$, $\phi_t : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ is a diffeomorphism, in the sense discussed above. Conversely, for a fixed point $p \in \widetilde{\mathcal{M}}$, we get a curve

$$\phi_t|_p : \mathbb{R} \to \widetilde{\mathcal{M}} \ ,$$

parametrized by t, which is called an *orbit* of ϕ_t and which passes through p at $t = 0$. Hence, to each orbit of ϕ_t in p we can associate a vector $\boldsymbol{w}|_p$, tangent to $\phi_t|_p$ at $t = 0$. Moving from p, we can describe the whole family ϕ_t with a vector field $\boldsymbol{w}$, which is called the *generator* of the transformation associated with the diffeomorphism.

Let us consider now the one-parameter family of pullbacks ϕ_t^* associated with ϕ_t. Infinitesimally, the action of ϕ_t^* on the metric tensor can be formalized through the notion of *Lie derivative*, which is defined as

$$\pounds_{\boldsymbol{w}}\tilde{g}_{ab} \doteq \lim_{t\to 0}\frac{1}{t}\left(\phi^*_{-t}\tilde{g}_{ab} - \tilde{g}_{ab}\right) \ ,$$

thus giving [48, App. C]

$$\pounds_{\boldsymbol{w}}\tilde{g}_{ab} = \widetilde{\nabla}_a w_b + \widetilde{\nabla}_b w_a \ , \tag{2.17}$$

with $\boldsymbol{w}$ the vector field that generates the transformation.

From a field theoretical point of view, the freedom brought by ϕ is encoded as a gauge symmetry of (2.15). The linearization procedure preserves this invariance, which is inherited by (2.16) as the gauge freedom carried by *linearized diffeomorphisms*. Indeed, given a vector field $\boldsymbol{w}$, the action (2.16) is invariant under the following gauge transformation, defined with respect to the background metric

$$\delta_{\boldsymbol{w}}\gamma_{ab} = -\pounds_{\boldsymbol{w}} g_{ab} \ . \tag{2.18}$$

By substitution of (2.17), linearized with respect to the background metric $\boldsymbol{g}$, we can explicitly write (2.18) in terms of the generator of the transformation

$$\gamma_{ab} \mapsto \gamma'_{ab} = \gamma_{ab} - 2\nabla_{(a}w_{b)} \ . \tag{2.19}$$

To put $\boldsymbol{L}$ in a more useful form, we shall exploit the freedom carried by (2.18), by imposing some suitable gauge condition. Since the theory we are studying is linear in the equations of motion, we do not have to deal with the typical problem involved with the introduction of gauge-breaking terms. Instead we gauge-fix the equation of motions, discussing the other possibility during the quantization procedure, by means of the BRST procedure.

Employing the redefinition of the perturbation field given by (2.12), we impose the *de Donder gauge condition* [18], which in some sense is related to the Lorentz gauge of the Maxwell theory [42]

$$\nabla^a\gamma_{ab} - \tfrac{1}{2}\nabla_b\gamma^a{}_a = 0 \ . \tag{2.20}$$

Under the field redefinition (2.12) the de Donder prescription reads

$$\nabla^a\bar{\gamma}_{ab} = 0 \ . \tag{2.21}$$

Before continuing our discussion we notice that, given a (k,l) tensor

$$\begin{aligned}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\, T^{\lambda_1\ldots\lambda_k}{}_{\sigma_1\ldots\sigma_l} = &-\sum_i R_{\mu\nu\rho}{}^{\lambda_i} T^{\lambda_1\ldots\rho\ldots\lambda_k}{}_{\sigma_1\ldots\sigma_l} \\ &+\sum_i R_{\mu\nu\sigma_i}{}^{\rho} T^{\lambda_1\ldots\lambda_k}{}_{\sigma_1\ldots\rho\ldots\sigma_l} \ .\end{aligned} \tag{2.22}$$

We simplify the equation of motion specializing (2.22) to $(1,1)$ tensors, thus leading to

$$2\nabla_{[\alpha}\nabla_{\mu]}\bar{\gamma}_\nu{}^\alpha = -R_{\alpha\mu\beta}{}^\alpha\bar{\gamma}_\nu{}^\beta + R_{\alpha\mu\nu}{}^\beta\bar{\gamma}_\beta{}^\alpha = R_{\alpha\mu\nu}{}^\beta\bar{\gamma}_\beta{}^\alpha \ , \tag{2.23}$$

for a vacuum spacetime background. In addition to the gauge condition (2.21), (2.23) gives that

$$\nabla_\alpha\nabla_{(\mu}\bar{\gamma}_{\nu)}{}^\alpha = \tfrac{1}{2}(R_{\alpha\mu\nu}{}^\beta\bar{\gamma}_\beta{}^\alpha + R_{\alpha\nu\mu}{}^\beta\bar{\gamma}_\beta{}^\alpha) = R_{\alpha\mu\nu\beta}\bar{\gamma}^{\beta\alpha} \ . \tag{2.24}$$

We substitute (2.24) in (2.13). Thus, the equations of motion (2.14) for the redefined perturbation (2.12) becomes

$$\Box\bar{\gamma}_{\mu\nu} - 2R_{\alpha\mu\nu\beta}\bar{\gamma}^{\alpha\beta} = 0 \ , \tag{2.25}$$

where we have defined the curved wave operator as $\Box = g^{ab}\nabla_a\nabla_b$. Equation (2.25) is often called the *Lichnerowicz tensor equation* for linearized gravity [30].

The flat limit It should be noticed that when we consider a Minkowski background, where $R_{\alpha\mu\nu\beta} = 0$, the de Donder gauge condition reduces to the Lorentz one

$$\partial^{\mu}\bar{\gamma}_{\mu\nu} = 0 \ , \tag{2.26}$$

while (2.25) reads

$$\Box\bar{\gamma}_{\mu\nu} = 0 \ ,$$

which is the flat waves equation as reported in [48, Ch. 4], describing the dynamics of a free perturbation on a flat spacetime, namely the propagation of a gravitational wave.

Again, the equation of motion (2.25) can be linked to the action of a tensor differential operator on $\boldsymbol{\gamma}$, whose components are given by

$$P_{\mu\nu}{}^{\alpha\beta} \doteq \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\Box - 2R^{\alpha}{}_{\mu\nu}{}^{\beta}, \quad P_{\mu\nu}{}^{\alpha\beta}\bar{\gamma}_{\alpha\beta} = 0 \ . \tag{2.27}$$

Through some work we have shown that it is possible to put the equation of motion (2.25) in a normally hyperbolic form. In the following section, we will discuss the Cauchy problem related (2.25), from the existence and uniqueness of the solutions to the difficulties in the construction of the advanced and retarded propagators of (2.27).

Generalized de Donder condition Before ending this section, we consider a generalization of the de Donder condition (2.20), which will be useful during the discussion of BRST quantization [17, 16]. We start again from the equation of motion of a free gravitational perturbation on a curved spacetime background

$$\nabla_{\alpha}\nabla_{(\mu}\gamma_{\nu)}{}^{\alpha} - \frac{1}{2}\nabla_{\mu}\nabla_{\nu}\gamma - \frac{1}{2}\Box\gamma_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box\gamma - \frac{1}{2}g_{\mu\nu}\nabla_{\alpha}\nabla_{\beta}\gamma^{\alpha\beta} = 0 \ . \tag{2.28}$$

Let us consider the following gauge condition

$$\nabla^{a}\gamma_{ab} - k\nabla_{b}\gamma^{a}{}_{a} = 0 \ , \tag{2.29}$$

which will be called *generalized de Donder gauge condition*. By following the same argument already used in (2.24), we substitute (2.29), thus getting

$$\nabla_{\alpha}\nabla_{(\mu}\gamma_{\nu)}{}^{\alpha} = k\nabla_{\mu}\nabla_{\nu}\gamma + R_{\alpha\mu\nu\beta}\gamma^{\beta\alpha} \ .$$

By substitution in (2.28), we get

$$(2k-1)\,\nabla_{\mu}\nabla_{\nu}\gamma + 2R_{\alpha\mu\nu\beta}\gamma^{\beta\alpha} - \Box\gamma_{\mu\nu} - (k-1)g_{\mu\nu}\Box\gamma = 0 \ . \tag{2.30}$$

We trace this equation, by multiplication of the inverse background metric, thus obtaining, for a vacuum spacetime background

$$(k-1)\Box\gamma = 0 \ .$$

Finally, by further substitution in (2.30), we obtain

$$(2k-1)\,\nabla_{\mu}\nabla_{\nu}\gamma + 2R_{\alpha\mu\nu\beta}\gamma^{\beta\alpha} - \Box\gamma_{\mu\nu} = 0 \ .$$

From this last result it becomes clear that, for the linearized theory, the case provided by $k = \frac{1}{2}$ is the only one which allows a normally hyperbolic set of equations of motion. Since this requirement is pivotal to the existence and uniqueness of the advanced and retarded propagator, we will always assume this condition for our further discussions.

2.3.2 Existence and uniqueness of solutions

In this section we discuss whether the Cauchy problem of linearized gravity on curved spacetimes is well-posed, or not. In the algebraic formulation of quantum field theory this is a main topic, which has been treated several times in the literature, both for the gravitational case [18, 24] and for other field theories [4, 3, 45].

A well-behavioured Cauchy problem is a must-have for any field theory, since it is the first step in the construction of different fundamental objects: from the classical description given by propagators to the algebraic structure of the quantum theory. Contrary to the Klein-Gordon theory, the case of a gravitational perturbation requires more work on a mathematical ground.

We start our discussion by recalling some definition, allowing for the description of a classical field ψ without any particular requirement on its geometrical properties. Let us consider a *vector bundle* $F(\mathcal{M}, \pi, V)$, i.e a fiber bundle with a vector space structure [4, Def. 2.1], [37, Ch. 9]. Here $\mathcal{M}$ denotes the base space, V the typical fiber and $\pi : F \to \mathcal{M}$ the projection function, such that each fiber $F_p \doteq \pi^{-1}(p)$ is isomorphic to the vector space V.

1. We call *section* any smooth inverse ψ of the projection map π. We denote $\Gamma(F)$ the *space of smooth sections* of F.

2. Let us consider $F_1(\mathcal{M}, \pi_1, V_1)$, $F_2(\mathcal{M}, \pi_2, V_2)$ two vector bundles with same base space $\mathcal{M}$. The *tensor product bundle* can be obtained by assigning a tensor product of fibers $V_1 \otimes V_2$ to any point $p \in \mathcal{M}$.

From a physical point of view, $\Gamma(F)$ can be interpreted as the *space of configuration field*, whose element ψ are *classical fields*. Moreover, different choices of the vector space V lead to different kind of field theories. For instance, $V = \mathbb{R}$ reduces to the case of a Klein-Gordon field, while $V = \mathbb{R}^4$ describes the structure of a spin-1 field.

We end this preamble by recalling that, given ψ a smooth section of F, a linear partial differential operator $L : \Gamma(F) \to \Gamma(F)$ on a globally hyperbolic spacetime $(\mathcal{M}, \boldsymbol{g})$, is called *normally hyperbolic* if it can be expressed as [4]

$$L\psi = -g^{\mu\nu} I_V \partial_\mu \partial_\nu \psi + A^\mu \partial_\mu \psi + B\psi \ ,$$

for every coordinate frame on $\mathcal{M}$, with A^μ and B smooth and I_V the identity on V.

Let us specialize this discussion to the case of a metric perturbation of the background spacetime $(\mathcal{M}, \boldsymbol{g})$. From the beginning of this chapter, $\bar{\boldsymbol{\gamma}}$ has been built through (2.1) as a smooth symmetric $(0, 2)$ tensor field on $\mathcal{M}$, which acts on

$$T\mathcal{M} \otimes T\mathcal{M} \ .$$

Let us denote $\mathcal{V} \doteq \mathrm{Sym}(T\mathcal{M}^* \otimes T\mathcal{M}^*)$ the symmetric tensor bundle, with base space $\mathcal{M}$ and $\mathrm{Sym}(T_p\mathcal{M}^* \otimes T_p\mathcal{M}^*)$ the typical fiber on $p \in \mathcal{M}$. Hence, the configuration space of gravitational perturbation is given by $\Gamma(\mathcal{V})$, with $\bar{\boldsymbol{\gamma}}(x)$ being any smooth section of $\mathcal{V}$.

We set up the necessary formalism of algebraic quantum field theory by smearing the linearized perturbation field $\boldsymbol{\gamma}$ with a *test tensor* $\boldsymbol{f}$, thus defining the notion of a *classical smeared field*

$$F_{\boldsymbol{f}}(\boldsymbol{\gamma}) \doteq \int \bar{\gamma}_{ab}(x) f^{ab}(x) \sqrt{-g} d^4x \ . \tag{2.31}$$

From now on, we shall commit a slight abuse of notation by adopting the prescription $F_{\boldsymbol{f}}(\gamma) \doteq \gamma(\boldsymbol{f})$. Moreover, the following discussion does not depend from the trace subtraction (2.12), which can be always inverted [24], giving

$$\bar{\gamma}(\boldsymbol{f}) = \gamma(\boldsymbol{f}) - \frac{1}{2}\int \gamma^a{}_a f^b{}_b \sqrt{-g} d^4x \ .$$

From a physical point of view this definition is actually justified from the following observation. Usually a (configuration) field permeates the spacetime by carrying the physical property of a particularly model. However, any experimental setup, namely a detector, is of finite spatial extent on $\mathcal{M}$, providing also a measure which belongs to a limited time interval. Prescription (2.31) reproduces this situation, making use of the test function to model the physical response of the detector (localized in $\operatorname{supp} f$), which gives a measure by actually smearing the configuration field $\bar{\gamma}_{ab}(x)$.

The test tensor $\boldsymbol{f}$, which lives in the *space of compactly supported sections* $\Gamma_0(\mathcal{V})$, will play an important role in the discussion of gauge-invariant quantization, whose treatment is delayed to the algebraic quantization chapter. Moreover, we would like to restrict our attention to those $\bar{\gamma}(f)$ which are formed by actually smearing solutions of the field equation (2.25). Indeed, we call *space of on-shell configurations*, the subspace of smooth sections $\Gamma(\mathcal{V})$ that contains the solutions to the field equation (2.25)

$$\mathrm{Sol}(\mathcal{V}) \doteq \{\gamma \in \Gamma(\mathcal{V}) \mid P_{ab}{}^{cd}\bar{\gamma}_{cd} = 0\} = \mathrm{Ker}(P) \ .$$

Now we have all the necessary instruments to solve the initial value problem associated with the equation of motion (2.25). As stressed before, several statement of this theorem can be found on [4, 3, 18, 24].

Theorem 1. *Let $(\mathcal{M}, \boldsymbol{g})$ a globally hyperbolic spacetime, with Σ a spacelike Cauchy surface and $\boldsymbol{n}$ its future-pointing unit normal vector field. Let $\boldsymbol{P} : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$ a normally hyperbolic tensor operator, whose components are given by (2.27). Let $(\boldsymbol{u}, \boldsymbol{v})$ a suitable choice of the initial data on Σ. The Cauchy problem given by*

$$\begin{cases} P_{ab}{}^{cd}\bar{\gamma}_{cd} = 0 \ , \\ \bar{\gamma}_{ab}|_\Sigma = u_{ab} \ , \\ (\nabla_n \bar{\gamma})_{ab}|_\Sigma = v_{ab} \ , \end{cases}$$

admits a unique solution $\bar{\boldsymbol{\gamma}} \in \mathrm{Sol}(\mathcal{V})$.

Once that the existence and uniqueness of the solutions of $\boldsymbol{P}$ is established, we can invert redefinition (2.12) to extend theorem 1 to the $\boldsymbol{L}$ operator [24].

2.3.3 The causal propagator

In the previous section we have showed that the dynamics of a gravitational perturbation can be uniquely solved by means of an initial value problem on a curved spacetime. In this section we discuss the role of propagators, and their multi-purpose importance: from the construction of the solutions to the quantization process.

As stressed before, the existence and uniqueness of solutions of $\boldsymbol{L}$ comes from the fact that the differential operator $\boldsymbol{P}$ is normally hyperbolic. Such a differential

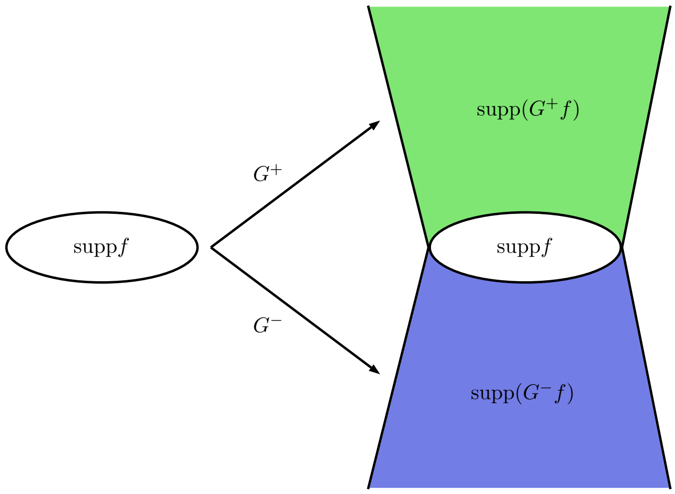


Figure 2.1: Advanced and retarded propagators and supp f

operator also admits a unique set of advanced (+) and retarded propagators (−) [18, 24], [3, Ch. 3], i.e the fundamental solutions of the equation of motion (2.25)

$$\boldsymbol{G}^{\pm} : \Gamma_0(\mathcal{V}) \longrightarrow \Gamma(\mathcal{V}), \quad \boldsymbol{G}^{\pm}\boldsymbol{L} = \boldsymbol{L}\boldsymbol{G}^{\pm} = \boldsymbol{I} \ , \tag{2.32}$$

which satisfies

$$\mathrm{supp}(\boldsymbol{G}^{\pm}\boldsymbol{f}) \subset J^{\pm}_{\mathcal{M}}(\mathrm{supp}(\boldsymbol{f})) \ , \tag{2.33}$$

with $\mathrm{supp}(\boldsymbol{f})$ the compact support of a test tensor $\boldsymbol{f} \in \Gamma_0(\mathcal{V})$. Hence, from (2.32) it follows that $\boldsymbol{L}(G^{\pm}f) = \boldsymbol{f}$.

As showed by (2.33), by applying $\boldsymbol{G}^{\pm}$ to a test tensor we generate a configuration field, i.e an element of $\Gamma(\mathcal{V})$, whose support is actually extended in the past or future light cone of $\mathrm{supp}(\boldsymbol{f})$. We sketch this situation in figure 2.1.

Explicitly, the action of the advanced and retarded propagators on $\Gamma_0(\mathcal{V})$ can be written in components by means of their distributional kernel

$$(G^{\pm ab}{}_{c'd'}f^{c'd'})(x) = \int G^{\pm ab}{}_{c'd'}(x,x')f^{c'd'}(x')\sqrt{-g}d^4x'$$

with x, x' labelling two points of the spacetime. Here, we are actually using the primed prescription to denote those indices which referred do primed coordinates on the spacetime.

The invertibility condition of $\boldsymbol{L}$ (2.32) actually reads on vacuum spacetimes [34]

$$(\Box g_{ac}g_{bd} - 2R_{cabd})G^{\pm ab}{}_{c'd'}(x,x') = g_{c'(c}g_{d)d'}\delta^4(x-x') \ .$$

From this last condition we should note that the expression of the propagators, *strongly* depends on the choice of the background spacetime, as well for everything that concern the dynamical evolution of the theory.

Once that the advanced and retarded solutions $\boldsymbol{G}^{\pm}$ are known, we can define the *causal propagator* as

$$G_{abc'd'} \doteq G^{-}_{abc'd'} - G^{+}_{abc'd'} \ .$$

From (2.32) and (2.33), it follows that

$$\boldsymbol{L}(\boldsymbol{G}\boldsymbol{f}) = 0 \ , \quad \operatorname{supp}(\boldsymbol{G}\boldsymbol{f}) \subset J_{\mathcal{M}}(\operatorname{supp}(\boldsymbol{f})) \ .$$

This last prescription suggests that: to any test tensor $\boldsymbol{f}$ we can associate a perturbation field $\boldsymbol{\gamma}(x) = (\boldsymbol{G}\boldsymbol{f})(x)$, which is a solution to (2.25). To give a complete characterization of $\mathrm{Sol}(\mathcal{M})$, we observe that, however, on-shell fields are not faithfully labelled by the test tensor. To show this, let us consider a test tensor $h^{ab} = L^{ab}{}_{cd} f^{cd}$, then the smeared field reads

$$\gamma(\boldsymbol{h}) = \int \gamma_{ab}(x)(Lf)^{ab}(x)\sqrt{-g}d^4x = \int (L\gamma)_{ab}(x) f^{ab}(x)\sqrt{-g}d^4x = 0 \ ,$$

which vanish for $\boldsymbol{\gamma} \in \mathrm{Sol}(\mathcal{M})$. Hence, any on-shell field vanishes when smeared on the image of $\boldsymbol{L}$, then $\gamma(\boldsymbol{f} + \boldsymbol{L}\boldsymbol{g}) = \gamma(\boldsymbol{f})$. To reduce this ambiguity, we can modify our initial prescription (2.31) by smearing against an equivalence class of test tensor

$$[\boldsymbol{f}] \doteq \{\boldsymbol{f} + \boldsymbol{L}\boldsymbol{g} \mid \boldsymbol{f}, \boldsymbol{g} \in \Gamma_0(\mathcal{M})\} \ .$$

This discussion makes clear that the causal propagator $\boldsymbol{G}$ provides an isomorphism which characterizes the space of on-sheel configuration

$$\mathrm{Sol}(\mathcal{M}) \simeq \Gamma_0(\mathcal{M})/\boldsymbol{L}(\Gamma_0(\mathcal{M})) \ ,$$

with $\boldsymbol{L}(\Gamma_0(\mathcal{M}))$ the image of $\boldsymbol{L}$ on the space of test tensors.

2.3.4 Degrees of freedom for curved gravitational waves

In the last section we impose the de Donder gauge condition (2.21), allowing the simplification of the equations of motion, which read (2.25). Nevertheless, as we are going to discuss, our choice did not completely fix the invariance carried by (2.19), which actually shows a residual freedom.

First, we consider the prescription of linearized diffeomorphisms (2.19) applied to the redefined perturbation field $\bar{\boldsymbol{\gamma}}$. Combining the trace subtraction (2.12) with (2.19), we compute the gauge transformation with respect to $\bar{\boldsymbol{\gamma}}$, getting

$$\bar{\gamma}_{ab} \mapsto \bar{\gamma}'_{ab} = \bar{\gamma}_{ab} - 2\nabla_{(a} w_{b)} + g_{ab}\nabla_c w^c \ .$$

Lowering the indices and taking the divergence of both members, we obtain that

$$\nabla^a \bar{\gamma}'_{ab} = \nabla^a \bar{\gamma}_{ab} - \Box w_b - 2\nabla_{[a}\nabla_{b]} w^a \ . \tag{2.34}$$

Moreover, we use property (2.22) of the Riemann tensor together with the background vacuum Einstein equation (1.6)

$$2\nabla_{[a}\nabla_{b]} w^a = -R_{ab\lambda}{}^{b} w^\lambda = -R_{a\lambda} w^\lambda = 0 \ . \tag{2.35}$$

Putting the last equation into (2.34) and imposing the de Donder gauge (2.21), we have that $\nabla^a \bar{\gamma}'_{ab} = 0$ if and only if

$$\Box w^a = 0 \ . \tag{2.36}$$

Hence, even working with fields that satisfy the de Donder condition, there exists a residual freedom, which is preserved by restricting the set of all gauge transformations $\delta_{\boldsymbol{w}}\bar{\gamma}$ to those satisfying (2.36). In other words, the choice (2.21) does not fix all the degrees of freedom carried by linearized diffeomorphisms. The residual gauge comes from the fact that (2.36) has an infinite number of solutions. However, equation (2.36) is normally hyperbolic, ensuring the existence of a unique solution $\boldsymbol{w}$, upon a suitable choice of the initial data (in the sense discussed by section 2.3.2) [42, 4]

$$\begin{cases} \Box w^a = 0 \ , \\ w^a|_\Sigma = u^a \ , \\ (\nabla_n w)^a|_\Sigma = v^a \ , \end{cases} \tag{2.37}$$

with Σ a spacelike Cauchy surface.

We can summarize the previous discussion by stating that this residual gauge can be fixed by taking additional conditions on the restriction of the gauge generator $\boldsymbol{w}$ to Σ. However, there is no guarantee that the desired condition can propagate at any time on $\mathcal{M}$. Precisely, this is the case of the TT gauge (transverse-traceless), which is given by

$$g^{ab}\gamma_{ab}|_\Sigma = 0 \ , \tag{2.38}$$

set up by requiring $\nabla_a w^a|_\Sigma = -\frac{1}{2}\gamma_{ab}$. On Schwarzschild spacetime for instance, the propagation of (2.38) is known to be afflicted by some topological obstruction, which can be avoided by requiring additional condition on $\gamma(\boldsymbol{f})$ [5, 18].

The knowledge of the de Donder condition (2.21), together with the residual one (2.37), allow us to perform a simple count of the degrees of freedom of a gravitational wave, propagating on curved spacetime background. Indeed

$$\#\text{dof of } \gamma_{ab} \quad : \quad \underbrace{16}_{(0,2)\text{ tensor}} \quad - \quad \underbrace{6}_{\text{index symmetry}} \quad - \quad \underbrace{4}_{\text{de Donder gauge}} \quad - \quad \underbrace{4}_{\text{residual gauge}} \quad = \quad 2$$

This simple calculation highlights what we expect from a physical point of view. Even on backgrounds described by a curved spacetime, gravitational waves propagates with two independent normal modes.

We complete this chapter by outlining that the explicit computation of the propagators (2.32) is not easy to achieve on a curved background. Indeed the calculation of $\boldsymbol{G}^\pm$ is related to the inversion of $\boldsymbol{L}$, which comes by studying the equation of motion (2.25). However, the Christoffel symbols both contained in $\Box\bar{\gamma}_{ab}$ and $R_{acdb}\bar{\gamma}^{cd}$ couples together the degrees of freedom carried by $\boldsymbol{\gamma}$, leading to a system of coupled second order partial differential equations. Nevertheless, a recent result have shown that on Schwarzschild, the Lichnerowicz equation (2.25) can be partially decoupled in two systems in upper triangular form, actually simplifying the computation of $\boldsymbol{G}^\pm$ [30].

Chapter 3

Quantum theory

3.1 Introduction

In the previous chapter we have developed all the instruments necessary to formulate the classical theory of linearized gravity. Here we are going to discuss the quantization of the theory, following the algebraic approach. Contrary to the scalar field theory, discussed in appendix A, the quantization of linearized gravity requires additional care due to the presence of gauge invariance. We will treat this problem by discussing two different point of view. On one hand, linearity of the equations of motion (2.25) allows one to give a quantization scheme without introducing any gauge-breaking term, recovering the freedom by the suitable choice of an equivalence class of configurational field [18, 4]. On the other hand, we will argue that the development of objects at least quadratic in $\boldsymbol{\gamma}$, such as the stress-energy tensor, requires the addition of ghost fields, thus leading to the quantization via BRS approach.

Once that the algebraic structure of the space of observables is known, we will deal with the problem of the computation of quantum expectation values, which will require the construction of a quantum state ω. In particular, we will focus on the choice of physically admissible states, i.e those that satisfies the Hadamard property.

3.2 Quantization and vacuum states, from flat to curved spacetimes. A motivation to the algebraic approach

In this section we briefly summarize the standard approach to quantization of a spin-2 fields on a flat spacetime [32, 50], highlighting the critical issues that comes up when aiming for a generalization to curved backgrounds.

Let us consider the Minkowski spacetime $(\mathbb{R}^4, \boldsymbol{\eta})$ on which we consider a spin-2 real free field to describe the gravitational perturbation $\boldsymbol{h}$[1]. Under the harmonic gauge (2.26), the action of linearized gravity (2.16) reduces to

$$S_{flat} = \frac{1}{32\pi G} \int [\tfrac{1}{2} h^{ab} \Box h_{ab} - \tfrac{1}{4} h \Box h] d^4x \; . \tag{3.1}$$

In this brief introduction we do not discuss the presence of gauge breaking terms

[1] For this particular section, we have slightly changed notation from $\boldsymbol{\gamma}$ to $\boldsymbol{h}$ to simply distinguish the flat theory from the curved one.

within S_{flat}. Indeed, we postpone the treatment of gauge symmetry for quantum theories to the following sections, by adopting the algebraic framework.

The field equations can be obtained by computing the variational derivative of (3.1), which gives

$$\Box h_{ab} = 0 \ , \tag{3.2}$$

with the flat wave operator given by $\Box = \eta^{ab}\partial_a\partial_b$. Since Minkowski spacetime is symmetric with respect to spacetime translations, the solutions of the equations of motion (3.2) are usually studied in terms of their Fourier transform $\tilde{h}_{ab}(p)$, which gives

$$p^2\tilde{h}_{ab}(p) = 0 \ , \tag{3.3}$$

with $p^2 = -\omega_p^2 + |\vec{p}|^2$. The solutions of (3.3) are given as distributions of the form

$$\tilde{h}_{ab}(p) = \sum_{\sigma=-2}^{+2} f(p,\sigma)u_{ab}(p,\sigma)\delta(p^2) \ , \tag{3.4}$$

with $u_{ab}(p,\sigma)$ called polarization tensor and $\sigma \in \mathbb{Z}$ the spin index. Indeed, $h_{ab}(p)$ has support in the light-cone, whose structure is described by the relativistic dispersion relation

$$-\omega_p^2 + |\vec{p}|^2 = 0 \ . \tag{3.5}$$

Due to the present of the delta distribution, the double-cone given by (3.5) can be separated in an *upper* and *lower* branches, which are respectively given by $\omega_p \geq 0$ and $\omega_p < 0$. Following this observation, we can consider the decomposition of the field h_{ab} in two modes, of positive and negative frequencies [50]

$$h_{ab}(x) = h^+_{ab}(x) + h^-_{ab}(x) \ .$$

By inverting the Fourier transform (3.4), we get

$$\begin{aligned} h^+_{ab}(x) &= \int \frac{1}{(2\pi)^3}\frac{1}{2\omega_p}\sum_{\sigma=-2}^{+2} a(p,\sigma)(p,\sigma)e^{ip\cdot x}d^3p \ , \\ h^-_{ab}(x) &= \int \frac{1}{(2\pi)^3}\frac{1}{2\omega_p}\sum_{\sigma=-2}^{+2} a^+(p,\sigma)u_{ab}(p,\sigma)e^{-ip\cdot x}d^3p \ , \end{aligned} \tag{3.6}$$

with $a(p,\sigma)$ and $a^+(p,\sigma)$ the Fourier coefficients associated with the positive and negative frequencies decomposition of $f(p,\sigma)$.

The quantization of the theory is usually achieved by promoting h^+_{ab} and h^-_{ab} to operators on the *Fock space*

$$\hat{h}^+_{ab},\ \hat{h}^-_{ab} : \mathcal{F} \to \mathcal{F} \ , \quad \mathcal{F} = \oplus_n(\otimes_{j=1}^n \mathcal{H}) \ ,$$

with $\mathcal{H}$ the Hilbert space of single particle. Actually, the operatorial character of both $\hat{h}^+_{ab}$ and $\hat{h}^-_{ab}$ requires the interpretation of the Fourier coefficients of (3.6) in terms of the so-called *creation* and *annihilation operators*, which satisfy the *canonical commutation relations*

$$\begin{aligned} &[\hat{a}(p,\sigma), \hat{a}^+(q,\lambda)] = i\delta_{\sigma\lambda}\delta^3(p-q) \ , \\ &[\hat{a}(p,\sigma), \hat{a}(q,\lambda)] = 0 \ , \\ &[\hat{a}^+(p,\sigma), \hat{a}^+(q,\lambda)] = 0 \ , \end{aligned} \tag{3.7}$$

with $\hbar = 1$. This formalism allows the definition of the *vacuum state* in terms of a vector $|0\rangle \in \mathcal{F}$, such that $a(p)|0\rangle = 0$.

The previous description can also be extended to the case of a curved spacetime background $(\mathcal{M}, \boldsymbol{g})$, by means of the following procedure.

1. Choice of a local coordinate frame $\{x^a\}$ on $(\mathcal{M}, \boldsymbol{g})$.

2. Identification of a local time function $t(x)$, by selection of a suitable time-like Killing vector field $\boldsymbol{k}$, if that even exists.

3. Definition of the Fourier transform with respect to $t(x)$.

4. Computation of the positive and negative frequencies coefficients.

5. Choice of a suitable Hilbert space to $\mathscr{H}$ describe the one-particle structure of the theory, in term of the positive frequency solution.

6. Construction of the Fock space $\mathcal{F}$ from $\mathscr{H}$, by promoting the positive and negative frequencies coefficients to creation and annihilation operators.

7. Identification of the vacuum state by means of a vector of $\mathcal{F}$, annihilated by the annihilation operator.

However, this scheme leads to several critical issues. Above all, on curved backgrounds, the choice of the time function $t(x)$ is highly non-unique[2], leading to multiple definitions of the Fourier transform, and thus of the creation and annihilation operators. Moreover, the canonical commutation relations (3.7) appears in a non-covariant fashion. Finally, the entire construction from the initial choice of the local coordinate frame leads to a definition of the vacuum state which is not diffeomorphism invariant. This has a significant physical consequence: different observers may experience non-equivalent choices of the vacuum state or different results of a measure of the same observable, as pictured by figure 3.1.

We can bring back the problem of applying the standard formalism to curved spacetime to one critical issue: from the beginning of the previous discussion, we have *simultaneously* chosen the quantization rule of fields together with their representation on a Hilbert space. In this way, we have obtained a description which is not well-behaved on curved spacetimes, for which the former scheme is not universal at all.

In this chapter we will adopt the algebraic approach to quantum fields on curved spacetimes. Indeed, we shall discuss a framework where the quantization of the fields is split from the choice of their representation on Hilbert spaces. On one hand, we shall investigate the algebraic structure behind quantum fields from an abstract point of view, thus ensuring a covariant description which holds for *any* choice of the spacetime background. On the other hand, we will show that, on curved spacetimes, the representation of fields highly depends on the choice of the quantum state, whose construction will be discussed in details. Indeed, we will end this chapter recovering the standard formalism, by means of the so-called Gelfand–Naimark–Segal (GNS) construction.

Before ending this section we review the highlights of the algebraic framework.

[2] A non-trivial exception to this is represented by those spacetime which admits a *global* time-like vector field.

- Covariance.
- Quantization procedure in a background-independent fashion.
- Characterization of the freedom in the choice of the quantum (vacuum) state.
- Independence from the representation on Hilbert spaces.
- Recovery of the standard formalism by the GNS construction.

3.3 Algebraic quantization

In this section we review the algebraic quantization of the perturbation field $\boldsymbol{\gamma}$. This discussion will be devoted to the construction of linear observables, which do not require the addition of any gauge-breaking terms in the action (2.16) [18, 24, 7].

Before discussing covariant canonical quantization, we review the problem of gauge invariance, described in the previous chapter, translating it in a more powerful mathematical description. We recall the definition of a linear classical or quantum observable, as a configuration field, i.e a smooth section of $\Gamma(\mathcal{V})$, smeared with an equivalence class of compactly supported test tensors $[\boldsymbol{f}]$ (2.31). In order to put the equation of motion in a normally hyperbolic form (2.25), we have exploited the de Donder gauge condition, by requiring $\nabla^a \gamma_{ab} = 0$, and actually reducing the configuration space, which splits as [18]

$$\Gamma(\mathcal{V}) = \Gamma^{dD}(\mathcal{V}) + \mathcal{G}(\mathcal{V}) \ , \tag{3.8}$$

with $\Gamma^{dD}(\mathcal{V})$ the space of those configurations that satisfy the gauge-fixing condition, and

$$\mathcal{G}(\mathcal{V}) = \{ \pounds_{\boldsymbol{w}} \boldsymbol{g} \in \Gamma(\mathcal{V}) \} \ , \tag{3.9}$$

the space of pure gauge fields. Indeed, prescription (3.8) states that any perturbation field which do not satisfy the condition (2.21) is indeed made of a pure gauge term. A proof of this decoupling has been actually sketched in section 2.3.4, while discussing the existence of the residual gauge [18].

The first step towards the quantization of linearized gravity requires the construction of a phase space, on which observables takes value, together with the definition of a suitable symplectic structure σ. Then, it will be possible to define the action of Poisson bracket on $\boldsymbol{\gamma}$, which allows the quantization through covariant canonical prescription.

However, the gauge freedom carried by linearized diffeomorphisms and encoded by (3.8) introduce a degeneracy on σ, which requires additional care. Let us consider the action of a pre-symplectic, i.e degenerate symplectic form, which acts on two on-shell smeared fields $\boldsymbol{\gamma^1}, \boldsymbol{\gamma^2}$ [18]

$$\sigma(\boldsymbol{\gamma^1}, \boldsymbol{\gamma^2}) \doteq \int_\Sigma (\gamma^1_{ab} \pi^{ab}_2 - \gamma^2_{ab} \pi^{ab}_1) d\Sigma \ ,$$

with Σ a spacelike Cauchy surface and $\boldsymbol{\pi}$ the conjugate momentum, which can be defined from the action (2.16) as

$$\pi^{ab} = -\frac{n_c}{\sqrt{-g}} \frac{\delta S_{lg}}{\delta \nabla_a \gamma_{bc}} \ .$$

The degeneracy arises when considering a pure gauge configuration. Indeed, it can be shown that $\sigma(\boldsymbol{\gamma^1}, \pounds_{\boldsymbol{w}}\boldsymbol{g}) = 0$ [18], actually giving

$$\sigma(\boldsymbol{\gamma^1}, \boldsymbol{\gamma^2} + \pounds_w \boldsymbol{g}) = \sigma(\boldsymbol{\gamma^1}, \boldsymbol{\gamma^2}) \ ,$$

for any choice of the gauge transformation generator $\boldsymbol{w}$. As for section 2.3.2, we can remove this degeneracy by considering an equivalence class of on-shell perturbation

$$[\gamma] \doteq \{\gamma + \pounds_w \boldsymbol{g} \mid \gamma \in \mathrm{Sol}(\mathcal{M})\} \ .$$

Thus a non degenerate symplectic form is given by

$$\sigma([\boldsymbol{\gamma^1}], [\boldsymbol{\gamma^2}]) \doteq \int_\Sigma (\gamma^1_{ab}\pi_2^{ab} - \gamma^2_{ab}\pi_1^{ab}) d\Sigma \ , \tag{3.10}$$

with $\boldsymbol{\gamma^1}$ and $\boldsymbol{\gamma^2}$, two choices of suitable representatives. These considerations can also be extended to linear observables (2.31). Indeed the contribution of pure gauge smeared fields reads

$$\pounds_{\boldsymbol{w}}\boldsymbol{g}([\boldsymbol{f}]) = \int (\nabla_{(a} w_{b)}) f^{ab} \sqrt{-g} d^4x = -\int w^b (\nabla^a f_{(ab)}) \sqrt{-g} d^4x \ .$$

Being $\boldsymbol{w}$ an arbitrary vector field, we can argue that the contribution of pure gauge fields vanishes if and only $\boldsymbol{f}$ satisfies $\nabla^a f_{(ab)} = 0$. Hence, adopting the functional notation of $\gamma(\boldsymbol{f})$, to clarity extent, we get

$$F_{[\boldsymbol{f}]}(\gamma + \pounds_{\boldsymbol{w}}\boldsymbol{g}) = F_{[\boldsymbol{f}]}(\gamma) \ ,$$

which allows us to restrict the smearing operation to the equivalence class $[\boldsymbol{\gamma}]$. From (3.9), we can build the gauge-invariant *phase space* as

$$\mathscr{P}(\mathcal{V}) = \mathrm{Sol}(\mathcal{V})/\mathcal{G}(\mathcal{V}) \ .$$

Finally we can define *gauge-invariant **linear** smeared fields*, as functionals on the phase space

$$F_{[\boldsymbol{f}]} : \mathscr{P}(\mathcal{V}) \to \mathbb{C} \ , \quad F_{[\boldsymbol{f}]}([\gamma]) = \int \gamma_{ab}(x) f^{ab}(x) \sqrt{-g} d^4x \ ,$$

with $\nabla^a f_{(ab)} = 0$ and $\boldsymbol{\gamma}$ any on-shell representative of the equivalence class. Despite $\boldsymbol{\gamma}$ being real, F takes value on complex numbers, admitting the possibility to deal with complex test tensors. From now on, we simplify this prescription by actually denoting $F_{[\boldsymbol{f}]}([\gamma]) \doteq \gamma(\boldsymbol{f})$, actually omitting the equivalence notation.

Once that the symplectic structure is formed, it can be related to the action of the causal propagator $\boldsymbol{G}$ [18]. Indeed, given two test tensors $\boldsymbol{f}, \boldsymbol{h} \in \Gamma_0(\mathcal{V})$, we denote

$$G(\boldsymbol{f}, \boldsymbol{h}) \doteq -2 \int f^{(ab)} G_{ab}{}^{cd} (\bar{h}_s)_{cd} \sqrt{-g} d^4x \ , \tag{3.11}$$

with $(\bar{h}_s)_{ab} \doteq h_{(ab)} - \frac{1}{2} g_{ab} \operatorname{tr} h$. Given $\gamma(\boldsymbol{f}), \gamma(\boldsymbol{f'})$ two smeared fields of the same configuration $\boldsymbol{\gamma}$, definition (3.11) is related to the expression of the symplectic form (3.10) by [18]

$$G(\boldsymbol{f}, \boldsymbol{f'}) = 4\sigma(\boldsymbol{G}\bar{\boldsymbol{f}}_{\boldsymbol{s}}, \boldsymbol{G}\bar{\boldsymbol{f}}'_{\boldsymbol{s}}) \ . \tag{3.12}$$

Hence, using (3.11) we can actually fix the symplectic structure of Poisson brackets [18, Th. 4.10], by means of the following prescription

$$\{\gamma(\boldsymbol{f}), \gamma(\boldsymbol{f'})\}_P = G(\boldsymbol{f}, \boldsymbol{f'}) \ . \tag{3.13}$$

Once that the classical theory is known, the quantization can be performed, promoting the smeared fields to elements of an algebra, whose structure is described by the canonical commutation relation

$$[\hat{\gamma}(\boldsymbol{f}), \hat{\gamma}(\boldsymbol{f'})] = i\{\gamma(\boldsymbol{f}), \gamma(\boldsymbol{f'})\}_P \hat{\mathbb{I}} \ , \tag{3.14}$$

with $\hbar = 1$. This prescription together with (3.13) give the canonical commutation rule in a covariant fashion, by means of the causal propagator, which gives

$$[\hat{\gamma}(\boldsymbol{f}), \hat{\gamma}(\boldsymbol{f'})] = iG(\boldsymbol{f}, \boldsymbol{f'}) \ . \tag{3.15}$$

In order to relate (3.15) with the algebraic structure generated by the observables, we give the following definition [31, 22].

We call *unital $*$-algebra*, any algebra $\mathcal{A}$ with unit $\hat{\mathbb{I}}$, which is endowed of an operation $*$, called *involution*, such that for any $A, B \in \mathcal{A}$

$$(A^*)^* = A \ ,$$
$$(AB)^* = B^* A^* \ .$$

The involution $*$ plays a pivotal role in both the introduction of the notion of observable, as a self-adjoint element of $\mathcal{A}$, and in the positivity requirement of quantum states, as we shall discuss in the following section. On the other hand, we will show that the standard language of QFT can be recovered when considering the representation of the elements of $\mathcal{A}$ as operators on a Hilbert space $\mathcal{H}$, for which the $*$ operation is mapped into the Hermitian adjoint one $\dagger$.

We can resume the results of this section by means of the following statement. The Dirac prescription imposed by (3.14) gives rise to the *CCR algebra* $\mathcal{A}(\mathcal{M})$, a unital $*$-algebra generated by $\hat{\gamma}(\boldsymbol{f})$ and the unity $\hat{\mathbb{I}}$. Indeed any quantum field $\hat{\gamma}(\boldsymbol{f}) \in \mathcal{A}(\mathcal{M})$ satisfies the following property [31, 18, 22].

1. **Linearity**: $\hat{\gamma}(c_1\boldsymbol{f_1} + c_2\boldsymbol{f_2}) = c_1\hat{\gamma}(\boldsymbol{f_1}) + c_2\hat{\gamma}(\boldsymbol{f_2})$, with $c_1, c_2 \in \mathbb{C}$.

2. **Hermiticity**: $\hat{\gamma}(\boldsymbol{f})^* = \hat{\gamma}(\boldsymbol{f}^*)$.

3. **Symmetry**: $\hat{\gamma}(\boldsymbol{f}) = 0$, for any anti-symmetric $\boldsymbol{f} \in \Gamma_0(\mathcal{V})$.

4. **Field equations**: $\hat{\gamma}(\boldsymbol{L}\boldsymbol{f}) = 0$.

5. **Commutation relation**: $[\hat{\gamma}(\boldsymbol{f}), \hat{\gamma}(\boldsymbol{f'})] = iG(\boldsymbol{f}, \boldsymbol{f'})\hat{\mathbb{I}}$.

With the test tensors of $\Gamma_0(\mathcal{V})$ that satisfies $\nabla_a f^{(ab)} = 0$.

We call (gauge-invariant) *observable*, generated by linear fields, any element $O \in \mathcal{A}(\mathcal{M})$, written as a polynomials of the generators of the algebra [31]

$$\hat{O} = c_{(0)}\hat{\mathbb{I}} + \sum_{i_1} c_{(1)}^{i_1} \hat{\gamma}(\boldsymbol{f_{i_1}^{(1)}}) + ... + \sum_{i_1,\dots,i_n} c_{(n)}^{i_1\dots i_n} \hat{\gamma}(\boldsymbol{f_{i_1}^{(n)}}) \cdots \hat{\gamma}(\boldsymbol{f_{i_n}^{(n)}}) \ ,$$

with $c_{(k)}^{i_1 \dots i_n} \in \mathbb{C}$, which satisfies

$$\hat{O}^* = \hat{O} \ .$$

During this section we have discussed the quantization of linearized gravity. The development of the algebraic approach have provided a way to implement the canonical commutation relations in a covariant fashion. Moreover, gauge-invariance has been implemented a posteriori, by actually restricting to a specific class of test tensors. The construction of $\mathcal{A}(\mathcal{M})$ set up the necessary formalism apt to understand the algebraic structure generated by $\hat{\gamma}(\boldsymbol{f})$, without referring to any particular representation on any Hilbert space and providing a description which holds for any choice of the spacetime background. Nevertheless, the standard language of configuration fields can always be recovered. Indeed, given $\boldsymbol{\gamma}$ a perturbation field, the canonical quantization rule reads

$$[\hat{\gamma}_{ab}(x), \hat{\gamma}_{cd}(x')] = iG_{abcd}(x,x')\hat{\mathbb{I}} \ , \tag{3.16}$$

with G_{abcd} the distributional kernel of (3.12). From (3.16) we may note that the structure of the algebra $\mathcal{A}(\mathcal{M})$ is deeply influenced by the choice of the background spacetime $(\mathcal{M}, \boldsymbol{g})$. As we have already stressed in the previous section, this dependence is carried and encoded by the causal propagator $\boldsymbol{G}$, which is generated by the equations of motion (2.25).

3.3.1 Beyond gauge-invariant observables. A call for ghosts

In this section we consider an extension of the previously discussed approach, going beyond the notion of gauge-invariant observables, generated by linear fields. These objects will play a pivotal role in understanding the evaporation of a black hole, since it will be involved in the computation of the stress-energy tensor T_{ab}, which is quadratic in $\boldsymbol{\gamma}$ and its covariant derivatives.

From a geometrical point of view, there exists different ways to contract all the indices to square a symmetric tensor γ_{ab}, e.g $\gamma_a{}^b\gamma_{bc}$ or $\gamma^a{}_a\gamma^b{}_b$.

As a first example we consider the configuration field given by $\hat{\gamma}_a{}^b(x)\hat{\gamma}_{bc}(x')$. Indeed, we are interested to the case of its coinciding point limit

$$\lim_{x' \to x} \hat{\gamma}_a{}^b(x)\hat{\gamma}_{bc}(x') \ ,$$

that once smeared, as for the Klein-Gordon theory [31], gives rise to the following observable

$$\hat{O}^2(\hat{\boldsymbol{\gamma}}; \boldsymbol{f}) = \int \hat{\gamma}_a{}^b(x)\hat{\gamma}_{bc}(x')f^{(ac)}(x)\delta^4(x-x')\sqrt{-g}\sqrt{-g'}d^4xd^4x' \ . \tag{3.17}$$

Here we are restricting to the symmetric part of $\boldsymbol{f}$, since the anti-symmetric one of the coinciding point limit of $\hat{\gamma}_a{}^b\hat{\gamma}_{bc}$ is either zero classically or fixed by the commutation relation in the quantum theory (3.16).

Let us consider the effect of a gauge transformation, i.e a linearized diffeomorphism

$$\hat{\gamma}_{ab} \to \hat{\gamma}_{ab} + \delta\hat{\gamma}_{ab} \ , \tag{3.18}$$

with $\delta\hat{\gamma}_{ab} = -2\nabla_{(a}w_{b)}$. It should be noted that, at this level, we are considering the role of a classical gauge generator, i.e a simple vector field $\boldsymbol{w}$. In the following section, we will consider the introduction of a ghost field, by actually promoting $\boldsymbol{w}$ to a quantum field $\hat{\boldsymbol{c}}$. By substitution in (3.17), we consider

$$\hat{O}^2(\hat{\gamma}+\delta\hat{\gamma};\boldsymbol{f}) = \hat{O}^2(\hat{\gamma};\boldsymbol{f}) + \delta\hat{O}^2(\hat{\gamma};\boldsymbol{f}) \ .$$

Once again, $\hat{O}^2$ is gauge-invariant if and only if $\delta\hat{O}^2 = 0$. To exploit this requirement, we isolate the contributions coming from $\boldsymbol{w}$, getting

$$\delta\hat{O}^2(\hat{\gamma};\boldsymbol{f}) = \int [\hat{\gamma}_a{}^b(x)\delta\hat{\gamma}_{bc}(x')+\delta\hat{\gamma}_a{}^b(x)\hat{\gamma}_{bc}(x')] f^{(ac)}(x)\delta^4(x-x')\sqrt{-g}\sqrt{-g'}d^4x d^4x'$$

By substitution of $\delta\hat{\gamma}$, and exploiting the integration of the delta distribution, we get

$$\begin{aligned}\delta\hat{O}^2(\hat{\gamma};\boldsymbol{f}) &= -4\int \hat{\gamma}_a{}^b \nabla_{(b}w_{c)} f^{(ac)}\sqrt{-g}d^4x = \\ &= -2\int \hat{\gamma}_a{}^b\nabla_b w_c f^{(ac)}\sqrt{-g}d^4x - 2\int \hat{\gamma}_a{}^b\nabla_c w_b f^{(ac)}\sqrt{-g}d^4x \ .\end{aligned}$$

We integrate it by parts. Since supp($\boldsymbol{f}$) is compact and the integration is taken along the entire spacetime, the boundary terms can be neglected, thus giving

$$\delta\hat{O}^2(\hat{\gamma};\boldsymbol{f}) = +2\int w_c\nabla_b(\hat{\gamma}_a{}^b f^{(ac)})\sqrt{-g}d^4x + 2\int w_b\nabla_c(\hat{\gamma}_a{}^b f^{(ac)})\sqrt{-g}d^4x$$

Renaming the dummy indices, it follows that

$$\delta\hat{O}^2(\hat{\gamma};\boldsymbol{f}) = +2\int w_c\left[\nabla_b(\hat{\gamma}_a{}^b f^{(ac)}) + \nabla_b(\hat{\gamma}_a{}^c f^{(ab)})\right]\sqrt{-g}d^4x$$

Since $\boldsymbol{w}$ is an arbitrary vector field, the requirement $\delta\hat{O}^2 = 0$ gives the following condition

$$\nabla^b\hat{\gamma}_{ab} f^{(ac)} + \hat{\gamma}_{ab}\nabla^b f^{(ac)} + \nabla_b\hat{\gamma}_a{}^c f^{(ab)} = 0 \ , \tag{3.19}$$

being $\nabla_a f^{(ab)} = 0$. However, (3.19) is not trivially satisfied by test tensors, without considering any additional condition for $\hat{\gamma}_{ab}$.

Before discussing the previous result (3.19), we generalize it, by not restricting to any particular contraction of $\hat{\gamma}_{ab}\hat{\gamma}_{cd}$. To this extent, we consider the following squared observable

$$\hat{O}^2(\hat{\gamma};\boldsymbol{f}) = \int \hat{\gamma}_{ab}(x)\hat{\gamma}_{cd}(x') f^{(ac)(bd)}(x)\delta^4(x-x')\sqrt{-g}\sqrt{-g'}d^4x d^4x' \ . \tag{3.20}$$

Here, we have introduced a *generalized test tensor* which encodes all the possible contractions of the perturbation fields. For instance (3.20) reduces to the first example (3.17), by taking

$$f^{(ac)(bd)} = f^{(ac)}g^{bd} \ .$$

We proceed in the same way, by inserting (3.18) in (3.20) and isolating the first-order gauge contribution

$$\delta\hat{O}^2(\hat{\gamma};\boldsymbol{f}) = -4\int \hat{\gamma}_{ab}\nabla_{(c}w_{d)} f^{(ac)(bd)}\sqrt{-g}d^4x$$

Exploiting the symmetrization and integrating by parts, we get

$$\delta\hat{O}^2(\hat{\boldsymbol{\gamma}};\boldsymbol{f}) = +2\int w_d \nabla_c(\hat{\gamma}_{ab} f^{(ac)(bd)})\sqrt{-g}d^4x + 2\int w_c \nabla_d(\hat{\gamma}_{ab} f^{(ac)(bd)})\sqrt{-g}d^4x \ ,$$

renaming the dummy indices, it follows that

$$\delta\hat{O}^2(\hat{\boldsymbol{\gamma}};\boldsymbol{f}) = +2\int w_d \left[\nabla_c(\hat{\gamma}_{ab} f^{(ac)(bd)}) + \nabla_c(\hat{\gamma}_{ab} f^{(ad)(bc)})\right]\sqrt{-g}d^4x$$

Once again, $\boldsymbol{w}$ is an arbitrary vector field, hence the requirement $\delta\hat{O}^2 = 0$ gives the following condition

$$\nabla_c\hat{\gamma}_{ab}\left(f^{(ac)(bd)} + f^{(ad)(bc)}\right) + \hat{\gamma}_{ab}\nabla_c\left(f^{(ac)(bd)} + f^{(ad)(bc)}\right) = 0 \ . \tag{3.21}$$

A possible solution to (3.21) would be given by the requirement that the generalized test tensor is anti-symmetric with respect to its second and fourth indices, such that

$$f^{(ac)(bd)} = -f^{(ad)(bc)} \ . \tag{3.22}$$

However, this solution makes everything trivial, since under (3.22) the squared observable (3.20) always vanishes.

In this section we have showed that the approach given by [18] has a non-trivial extension to the case of squared gauge-invariant observables, for which the condition $\nabla_a f^{(ab)} = 0$ is not enough. Unless we restrict the space of configuration fields γ_{ab} to achieve (3.21), the construction of $\hat{O}^2$ and so on, still requires the introduction of ghost fields and the quantization via BRST approach.

3.4 BRST quantization

We discuss an alternative approach to treat the gauge freedom of linearized gravity, which comes up to be very useful when dealing with non-linear gauge-invariant observable. We will consider the role of gauge-breaking terms, introducing the notion of ghost field while reviewing the quantization procedure by means of the BRST approach.

We start from the action of linearized gravity [1]

$$\begin{aligned} S_{lg} = \frac{1}{32\pi G}\int [&\tfrac{1}{2}\gamma^{ab}\Box\gamma_{ab} - \tfrac{1}{4}\gamma^a{}_a\Box\gamma^b{}_b + (\nabla^a\gamma_{ab} - \tfrac{1}{2}\nabla_b\gamma^a{}_a)^2 + \gamma^{ab}R_{acbd}\gamma^{cd} + \\ &+ \gamma^a{}_b R^{bc}\gamma_{ac} - \gamma^c{}_c\gamma^{ab}R_{ab} - \tfrac{1}{2}R\gamma_{ab}\gamma^{ab} + \tfrac{1}{4}R(\gamma^a{}_a)^2]\sqrt{-g}d^4x \ . \end{aligned} \tag{3.23}$$

As stressed before, this action is invariant with respect to gauge transformations given by linearized diffeomorphisms[3]

$$\delta_{\boldsymbol{w}}\gamma_{ab} = 2\nabla_{(a}w_{b)} \ . \tag{3.24}$$

We consider the generalized de Donder condition

$$G_b^{dD}(\gamma) \doteq \nabla^a\gamma_{ab} - k\nabla_b\gamma^a{}_a = 0 \ , \tag{3.25}$$

[3]Here we have slightly changed the prescription adopted in the previous sections, by absorbing the minus sign in the generator $\boldsymbol{w}$.

with $k \neq 1$. According the arguments of section 2.3, for the linearized theory we will always assume that $k = \frac{1}{2}$. However, we present the following discussion on a more general ground.

In order to impose it directly in the action (3.23), we introduce the following gauge-fixing term by means of a auxiliary field b^a [1, 17]

$$S_{gf} = \frac{1}{32\pi G}\int b^a[\nabla^b\gamma_{ab} - k\nabla_a\gamma^b{}_b + \tfrac{\alpha}{2}b_a]\sqrt{-g}d^4x \ , \tag{3.26}$$

with α a gauge parameter. This term can be obtained by squaring (3.25), to form a scalar object, i.e a candidate to be considered in the action. Indeed, (3.26) correctly reduces to (3.25) by imposing the equations motion of $\boldsymbol{b}$, which gives

$$S_{gf} = -\frac{1}{32\pi G}\int \frac{1}{2\alpha}[\nabla^a\gamma_{ab} - k\nabla_b\gamma^a{}_a][\nabla_c\gamma^{cb} - k\nabla^b\gamma^c{}_c]\sqrt{-g}d^4x \ .$$

In order to discuss a gauge-invariant quantization, while preserving the hyperbolic form of the field equations (2.25) (thus ensuring the existence of the causal propagator), we need to compensate the gauge-breaking term (3.26). This operation can be successfully achieved by replacing the original gauge-symmetry by a new one, called BRS symmetry.

The BRS transformation can be obtained by the identification of the classical gauge generator $\boldsymbol{w}$ with a new field, called the Fadeev-Popov ghost $\boldsymbol{c}$ [38, 17]. Hence (3.24), reads

$$\delta_{\boldsymbol{c}}\gamma_{ab} = 2\nabla_{(a}c_{b)} \ . \tag{3.27}$$

The ghost action can be actually derived by considering the effect of the BRS transformation (3.27) on (3.25). Then, one can form a scalar quantity by multiplication of this result with an anti-ghost field $\bar{\boldsymbol{c}}$, thus giving [17]

$$S_{gh} = -\frac{i}{32\pi G}\int \nabla^a\bar{c}^b[\nabla_a c_b + \nabla_b c_a - g_{ab}\nabla_d c^d]\sqrt{-g}d^4x \ , \tag{3.28}$$

which compensate the symmetry breaking induced by (3.26). It should be noted that the ghost $\boldsymbol{c}$ and the anti-ghost $\bar{\boldsymbol{c}}$ are two different fields, given [38]

$$(c^a)^* = c^a \ , \quad (\bar{c}^a)^* = \bar{c}^a \ .$$

By means of property (2.35), expression (3.28) actually reduces to

$$S_{gh} = \frac{i}{32\pi G}\int \bar{c}^a[g_{ab}\Box + R_{ab}]c^b\sqrt{-g}d^4x \ . \tag{3.29}$$

The linearized BRST transformation for the family $F = \{\boldsymbol{\gamma}, \boldsymbol{b}, \boldsymbol{c}, \bar{\boldsymbol{c}}\}$ is implemented by considering an operator s, acting on fields as an infinitesimal gauge transformation [16, 27]

$$\begin{aligned} s\gamma_{ab} &= (\nabla_a c_b + \nabla_b c_a) \ , \\ sb_a &= 0 \ , \\ s\bar{c}_a &= ib_a \ , \\ sc^a &= 0 \ , \end{aligned}$$

which satisfies for $\boldsymbol{X}, \boldsymbol{Y} \in F$ and $a, b \in \mathbb{C}$

$$\begin{aligned}
&s(a\boldsymbol{X} + b\boldsymbol{Y}) = as(\boldsymbol{X}) + bs(\boldsymbol{Y}) \ , \\
&s(\boldsymbol{X}\boldsymbol{Y}) = s(\boldsymbol{X})\boldsymbol{Y} + \epsilon(\boldsymbol{X})\boldsymbol{X}s(\boldsymbol{Y}) \ , \\
&s(\boldsymbol{X}^*) = \epsilon(\boldsymbol{X})[s(\boldsymbol{X})]^* \ , \\
&[s, \nabla_a] = 0 \ .
\end{aligned}$$

Here $\epsilon(\boldsymbol{X})$ is a number equal to 1 or -1, according to whether $\boldsymbol{X}$ represents a bosonic or fermionic field of the family F [38].

From the previous definition it follows immediately that the BRS operator is *nilpotent* [38, 16]

$$s^2 = 0 \ , \tag{3.30}$$

By definition, the action of linearized gravity is invariant with respect to the BRS-symmetry

$$s(S_{lg}) = 0 \ .$$

On the other hand, one can show that the gauge-fixing and ghost terms lies in the image of s [16]

$$S_{gf} + S_{gh} = -\frac{i}{32\pi G}\int s[\bar{c}^a(\nabla^b\gamma_{ab} - k\nabla_a{\gamma^b}_b + \tfrac{\alpha}{2}b_a)]\sqrt{-g}d^4x \ .$$

Hence, property (3.30) ensures that the introduction of ghost fields actually compensates the gauge-breaking terms, thus ensuring

$$s(S_{gf} + S_{gh}) = 0 \ .$$

Thus, the complete action is BRS-invariant

$$s(S_{lg} + S_{gf} + S_{gh}) = 0 \ .$$

By the computation of the variational derivative of (3.23), (3.26) and (3.29), we obtain the equations of motion for $\boldsymbol{\gamma}$ and $\boldsymbol{c}$. For a vacuum spacetime (1.6), we get

$$\Box\bar{\gamma}_{ab} - 2R_{cabd}\bar{\gamma}^{cd} - 2\nabla_{(a}\nabla^c\bar{\gamma}_{b)c} + g_{ab}\nabla^c\nabla^d\bar{\gamma}_{cd} - \nabla_{(a}b_{b)} + kg_{ab}\nabla^c b_c = 0 \ , \tag{3.31}$$

together with the constraint

$$b_a = -\tfrac{1}{\alpha}(\nabla^b\gamma_{ab} - k\nabla_a{\gamma^b}_b) \ , \tag{3.32}$$

while for ghost fields

$$\Box c_a = 0 \ , \quad \Box\bar{c}_a = 0 \ . \tag{3.33}$$

By substitution of (3.32) in (3.31), with gauge parameters $\alpha = k = \frac{1}{2}$, we obtain that

$$P_{ab}{}^{cd}\bar{\gamma}_{cd} = \Box\bar{\gamma}_{ab} - 2R_{cabd}\bar{\gamma}^{cd} = 0 \ . \tag{3.34}$$

Equations (3.34) and (3.33) are now normally hyperbolic, ensuring the existence and uniqueness of the solutions of the respective Cauchy problems, by the same argument of sections 2.3.2 and 2.3.3. It should be noticed that the BRS approach have led to the same equation of motion of linearized gravity (2.25), by exploiting the de Donder condition through the auxiliary field b^a while preserving the gauge symmetry with the introduction of ghost fields. Indeed, equations (3.31), (3.32) and (3.33) can be viewed in terms of a matrix of differential operators [7]. Let us consider a map $\mathcal{R}$, which associates to a tensor field $\boldsymbol{T}$ its redefined counterpart as in (2.12), such that

$$(\mathcal{R}T)_{ab} \doteq T_{ab} - \frac{1}{2} g_{ab} T^c{}_c \ . \tag{3.35}$$

Before going further, we review some useful properties of (3.35) [7].

1. From (3.35) we can immediately observe that $\mathcal{R}$ is idempotent, being $\mathcal{R}^2 = I$.
2. On vacuum spacetimes (3.34) implies that $\Box\gamma^a{}_a = \Box\bar{\gamma}^b{}_b = 0$, from which it follows that $\mathcal{R}$ actually commutes with the Lichnerowicz operator $P_{ab}{}^{cd}$.

Let us consider the following tensor differential operator

$$\mathcal{A}_{abcd} \doteq (g_{ac}g_{bd}\Box - 2R_{cabd} - 2g_{d(a}\nabla_{b)}\nabla_c + g_{ab}\nabla_c\nabla_d)\mathcal{R} \ ,$$

which acts on γ_{ab}, being actually involved in part of (3.31) and giving the equations of motion (2.28), for a redefined perturbation field $\bar{\gamma}_{ab}$. Hence, for $\alpha = k = \frac{1}{2}$ and by composition with (3.35), we can group the equations of motions for $F = \{\boldsymbol{\gamma}, \boldsymbol{b}, \boldsymbol{c}, \bar{\boldsymbol{c}}\}$ into a matrix of differential operators $\mathcal{K}$, such that

$$\mathcal{K}(F) \doteq \begin{bmatrix} \mathcal{A}_{ab}{}^{cd} & -\mathcal{R}g^c{}_{(a}\nabla_{b)} & 0 & 0 \\ \nabla^d\mathcal{R} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -i\Box \\ 0 & 0 & i\Box & 0 \end{bmatrix} \begin{bmatrix} \gamma_{cd} \\ b_c \\ c_a \\ \bar{c}_a \end{bmatrix} = 0 \ . \tag{3.36}$$

By direct computations we observe that

$$\mathcal{A}_{ab}{}^{cd} = P_{ab}{}^{cd}\mathcal{R} - 2\mathcal{R}g^c{}_{(a}\nabla_{b)}\nabla^d\mathcal{R} \ ,$$

which, by substitution in (3.36), gives

$$\begin{bmatrix} P_{ab}{}^{cd}\mathcal{R} - 2\mathcal{R}g^c{}_{(a}\nabla_{b)}\nabla^d\mathcal{R} & -\mathcal{R}g^c{}_{(a}\nabla_{b)} & 0 & 0 \\ \nabla^d\mathcal{R} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -i\Box \\ 0 & 0 & i\Box & 0 \end{bmatrix} \begin{bmatrix} \gamma_{cd} \\ b_c \\ c_a \\ \bar{c}_a \end{bmatrix} = 0 \ . \tag{3.37}$$

Since the equations of motion are normally hyperbolic, the theorem of section 2.3.3 and its counterpart for vector fields [3] guarantee the existence and uniqueness of the advanced and retarded solutions associated to the perturbation and ghost fields, respectively $G^{\pm}{}_{ab}{}^{c'd'}(x,x')$ and $E^{\pm}{}_a{}^b(x,x')$, as fundamental solutions of [34]

$$P_{ab}{}^{ef}G^{\pm}{}_{ef}{}^{c'd'}(x,x') = g_a{}^{(c'}g_b{}^{d')}\delta^4(x,x') \ ,$$
$$\Box E^{\pm}{}_a{}^{b'}(x,x') = g_a{}^{b'}\delta^4(x,x') \ ,$$

with the causality requirement such that, for any test tensor $\boldsymbol{f}$, the propagators $\boldsymbol{G}^-$, $\boldsymbol{E}^-$ and $\boldsymbol{G}^+$, $\boldsymbol{E}^+$ are respectively supported in the casual past and future of supp $\boldsymbol{f}$ (see figure 2.1 for a geometric intuition).

From the knowledge of the fundamental solutions, we can obtain the matrix expression of the propagators $\Delta^\pm(x,x')$ acting on F, by inverting the system of equations of motion (3.37), thus getting [7]

$$\Delta^\pm = \begin{bmatrix} \mathcal{R}G^{\pm}{}_{ab}{}^{c'd'} & +2\mathcal{R}G^{\pm}{}_{ab}{}^{d'e'}\mathcal{R}g^{c'}{}_{(d'}\nabla_{e')} & 0 & 0 \\ -2\nabla^b G^{\pm}{}_{ab}{}^{c'd'} & -4\nabla^b G^{\pm}{}_{ab}{}^{c'd'}\mathcal{R}g^{c'}{}_{(d'}\nabla_{e')} + 2g_a{}^{c'}\delta^4 & 0 & 0 \\ 0 & 0 & 0 & -iE^{\pm}{}_a{}^{b'} \\ 0 & 0 & iE^{\pm}{}_a{}^{b'} & 0 \end{bmatrix} .$$

From the knowledge of propagators, we can achieve the quantization of the theory by means of the algebraic approach, already described in the previous section. Indeed, we define the smeared perturbation and ghost fields as

$$\hat{\gamma}(\boldsymbol{f}) = \int \hat{\gamma}_{ab}(x) f^{ab}(x)\sqrt{-g}d^4x \ , \tag{3.38}$$

$$\hat{c}(\boldsymbol{h}) = \int \hat{c}_a(x) h^a(x)\sqrt{-g}d^4x \ , \tag{3.39}$$

$$\hat{\bar{c}}(\boldsymbol{h}) = \int \hat{\bar{c}}_a(x) h^a(x)\sqrt{-g}d^4x \ , \tag{3.40}$$

promoting them to be generators of a unital $*$-algebra $\mathcal{F}(\mathcal{M})$, with unity $\hat{\mathbb{I}}$, whose abstract structure is fixed by the following set of canonical commutation relations [27]

$$\begin{aligned} &[\hat{\gamma}(\boldsymbol{f}), \hat{\gamma}(\boldsymbol{f'})] = iG(\boldsymbol{f}, \boldsymbol{f'})\hat{\mathbb{I}} \ , \\ &\{\hat{c}(\boldsymbol{h}), \hat{\bar{c}}(\boldsymbol{h'})\} = iE(\boldsymbol{h}, \boldsymbol{h'})\hat{\mathbb{I}} \ , \\ &\{\hat{c}(\boldsymbol{h}), \hat{c}(\boldsymbol{h'})\} = \{\hat{\bar{c}}(\boldsymbol{h}), \hat{\bar{c}}(\boldsymbol{h'})\} = 0 \ , \end{aligned}$$

with $\boldsymbol{G}$ and $\boldsymbol{E}$ the perturbation and ghost causal propagators, respectively defined as $\boldsymbol{G} \doteq \boldsymbol{G}^- - \boldsymbol{G}^+$ and $\boldsymbol{E} \doteq \boldsymbol{E}^- - \boldsymbol{E}^+$. Moreover, $\{\cdot,\cdot\}$ denote the anti-commutator, whose action is given by

$$\{\hat{c}(\boldsymbol{h}), \hat{c}(\boldsymbol{h'})\} \doteq \hat{c}(\boldsymbol{h})\hat{c}(\boldsymbol{h'}) + \hat{c}(\boldsymbol{h'})\hat{c}(\boldsymbol{h}) \ .$$

On the other hand, since the auxiliary field b_a is related to the derivative of the perturbation field by (3.32), its algebraic properties can actually be derived from those of γ_{ab}.

Since the smeared fields (3.38), (3.39) and (3.40), together with $\hat{\mathbb{I}}$ are the generators of the algebra, the elements of $\mathcal{F}(\mathcal{M})$ are polynomials in $\hat{\gamma}$, $\hat{c}$, $\hat{\bar{c}}$ and their derivatives. However the ghost fields has been introduced as auxiliary objects with no physical meaning. Hence, it necessary to account for this observation by stating that (gauge-invariant) observables do not take contributions from the auxiliary fields, requiring to be given by those functional which are non-trivially s-invariant. Indeed we define the space of gauge-invariant observables as [14]

$$\mathscr{G}(\mathcal{M}) \doteq \mathrm{Ker}(s)/\,\mathrm{Im}(s) \tag{3.41}$$

3.5 Quantum states

In the previous sections we have firstly described $\mathcal{A}(\mathcal{M})$, which acts as the algebra of observables, which are gauge-invariant at linear order. Through the BRST approach it has been possible to go a little further, actually constructing gauge-invariant observables from polynomials of fields and ghosts, which generate the algebra $\mathcal{F}(\mathcal{M})$.

Let us consider an observable $\hat{O}$. We would like to link the abstract algebraic structure encoded by the commutation relations (3.16) with the physical reality, by predicting the results of an hypothetical measure of $\hat{O}$. Indeed, the bridge between any quantum theory and its experimental prediction is provided by the notion of quantum state, which allows the computation of the expectation value $\langle \hat{O} \rangle$.

We call *quantum state*, or simply state, any linear functional ω on the $*$-algebra of observables [31]

$$\omega : \mathcal{F}(\mathcal{M}) \to \mathbb{C} \; . \tag{3.42}$$

Through ω we associate to any observable in $\mathcal{F}(\mathcal{M})$ its *expectation value* $\langle \cdot \rangle_\omega = \omega(\cdot)$. Moreover, we say that ω is *normalized* if $\omega(\hat{\mathbb{I}}) = 1$, with $\hat{\mathbb{I}} \in \mathcal{F}(\mathcal{M})$.

The algebra $\mathcal{F}(\mathcal{M})$ is richer than $\mathcal{A}(\mathcal{M})$, since it also describes the contribution coming from ghost fields. Actually, the state ω may acts differently on $\hat{\gamma}$ and $\hat{c}$, as we shall discuss in details now.

Let us consider, as a first example, an element of $\mathcal{F}(\mathcal{M})$, built as a polynomial of $\hat{\gamma}$. We define the *n-point correlation function* of the state as a distribution acting on test tensors of $\Gamma_0(\mathcal{V})$

$$\omega_n(\boldsymbol{f_1}, \boldsymbol{f_2}, ..., \boldsymbol{f_n}) \doteq \omega(\hat{\gamma}(\boldsymbol{f_1})\hat{\gamma}(\boldsymbol{f_2}) \cdots \hat{\gamma}(\boldsymbol{f_n})) \; , \tag{3.43}$$

given by[4]

$$\omega_n(\boldsymbol{f_1}, \boldsymbol{f_2}, ..., \boldsymbol{f_n}) = \int \omega_{a_1 \dots a_{2n}}(x_1, x_2, ..., x_n) f_1^{a_1 a_2}(x_1) ... f_n^{a_{2n-1} a_{2n}}(x_n) dvolg_g^n \; ,$$

and with

$$dvolg_g^n = (-g(x_1))^{1/2} \cdots (-g(x_n))^{1/2} d^4 x_1 \cdots d^4 x_n \; .$$

A state ω is said *quasi-free* if its behaviour is fixed by the two-point correlation function $\omega_2(\boldsymbol{f_1}, \boldsymbol{f_2})$ [31], by means of the Wick theorem [46]

$$\begin{aligned} &\omega_n(\boldsymbol{f_1}, \boldsymbol{f_2}, ..., \boldsymbol{f_n}) = 0 \quad \text{for } n \text{ odd}, \\ &\omega_n(\boldsymbol{f_1}, \boldsymbol{f_2}, ..., \boldsymbol{f_n}) = \sum_{\text{part.}} \omega_2(\boldsymbol{f_{i_1}}, \boldsymbol{f_{i_2}}) \cdots \omega_2(\boldsymbol{f_{i_{n-1}}}, \boldsymbol{f_{i_n}}) \quad \text{for } n \text{ even}, \end{aligned}$$

with the sum to be taken among all the possible partitions of $\{1, 2, ..., n\}$. The notion of quasi-free states have an important physical interpretation in terms of those state which admits a representation of the algebra of observables in terms of the one-particle structure [22]. More details will be given in the following section.

The previous definition allows us to successfully study a quasi-free state by means of ω_2, whose distributional kernel can be written as

$$\omega_{abc'd'}(x, x') = \langle \hat{\gamma}_{ab}(x) \hat{\gamma}_{c'd'}(x') \rangle_\omega \; ,$$

[4] We drop the subscript for the integral kernel of ω_n, since the number of indexes is far enough to distinguish ω_2 from ω_4 and so on.

according to the notation fixed by (3.43). However, the choice of ω_2 is not totally arbitrary, since its anti-symmetric part is completely fixed by the canonical commutation relation

$$\omega_2(\boldsymbol{f}, \boldsymbol{f'}) - \omega_2(\boldsymbol{f'}, \boldsymbol{f}) = iG(\boldsymbol{f}, \boldsymbol{f'}) \ ,$$

while being a bi-solution of the equations of motion (2.25)

$$\omega_2(\boldsymbol{P}\bar{\boldsymbol{f}}, \boldsymbol{f'}) = \omega_2(\boldsymbol{f}, \boldsymbol{P}\bar{\boldsymbol{f}}') = 0 \ . \tag{3.44}$$

Once again, the geometrical properties of the background spacetime $(\mathcal{M}, \boldsymbol{g})$ together with field equations (2.25) are fundamental, since they both affect the computation of ω and the result of the expectation value of observables.

Mutatis mutandis, we can define the action of the state ω on the ghost fields contributions in terms of the two-point correlation function

$$\omega_{ab'}(x, x') = \langle \hat{c}_a(x)\hat{c}_b(x')\rangle_\omega \ . \tag{3.45}$$

Actually, this prescription satisfies the same properties of $\omega_{abc'd'}$, preserving the equations of motion on both its arguments and having its anti-symmetric part fixed by the causal propagator $\boldsymbol{E}$. Again, if the state is quasi-free, we can use the Wick theorem to completely describe its behaviour in terms of (3.45).

Since the equation of motions of $\hat{\gamma}_{ab}$ are not mixed with those of $\hat{c}_a$ and $\hat{\bar{c}}_a$ (3.36), we can assume that a quasi-free state ω shows two independent contributions, written in terms of the two-point functions $\omega_{abc'd'}$ and $\omega_{ab'}$. This observation allows us to write ω in the following matrix form [7]

$$\omega(F) = \begin{bmatrix} \mathcal{R}\omega_{abc'd'} & +2\mathcal{R}\omega_{abd'e'}\mathcal{R}g^{c'(d'}\nabla^{e')} & 0 & 0 \\ -2\nabla^b\omega_{abc'd'} & -4\nabla^b\omega_{abc'd'}\mathcal{R}g^{c'(d'}\nabla^{e')} & 0 & 0 \\ 0 & 0 & 0 & -i\omega_{ab'} \\ 0 & 0 & i\omega_{ab'} & 0 \end{bmatrix} , \tag{3.46}$$

with F the family of fields already considered in (3.36). For this very reason, we will separately discuss the construction of Hadamard states for the gravitational and ghost fields.

Before ending this section, we point out that, when dealing with gauge theories such as linearized gravity, the positivity requirement of states need to be dropped on $\mathcal{F}$. The reason for this choice can traced back to the presence of ghost fields, which are elements of $\mathcal{F}(\mathcal{M})$ actually contributing to states with negative norm. Nevertheless, positivity is restored when working with gauge invariant observables, namely on the cohomology of s (3.41), on which

$$\omega(O^*O) \geq 0 \quad \text{for any } O \in \mathscr{G}(\mathcal{M}) \tag{3.47}$$

Indeed, through (3.47) we are actually requiring that ω must be compatible with the definition of $\mathscr{G}(\mathcal{M})$, since unphysical fields like $\hat{c}$ cannot contribute to the expectation value $\langle O\rangle_\omega$ of a gauge-invariant observable.

3.5.1 Hadamard quasi-free states for linearized gravity

These sections are devoted to the discussion of Hadamard states and their properties for a gravitational perturbation field. Mutatis mutandis, the ghosts contribution will be separately reviewed in 3.5.5, since in any case it will play a pivotal role in establishing the role of the stress-energy tensor T_{ab} in black hole evaporation.

Let us discuss the physical property of ω_2, by first giving some definition. We call *bi-tensor*, any (h,l) tensor $\boldsymbol{T}$ in $(\mathcal{M},\boldsymbol{g})$ which depends upon two points of the spacetime x,x'. Indeed, we adopt the convention of [44, 22] denoting with a prime all the indices referred to the label x'. Moreover, we denote the coinciding point limit of $\boldsymbol{T}$ as

$$[T_{ab}] = \lim_{x'\to x} T_{ab'}(x,x')$$

We will take advantage of this formalism to better characterize the two-point correlation function $\omega_{ab'}(x,x')$.

Due to the distributional character of the configuration field $\gamma_{ab}(x)$, the integral kernel of ω_2 diverges in the coincide limit $[\omega_{ab'}]$. However, the presence of divergences of this kind allows us to give a criterion to select all the *physical admissible* states.

Let us consider a subset $\mathcal{O}\subset\mathcal{M}$. We say that $\mathcal{O}$ is *geodesically convex* if each two points $x,y\in\mathcal{O}$ are connected by a unique geodesic, which is completely contained in $\mathcal{O}$ [22].

Given any two point x, x' on a convex subspace $\mathcal{O}$, we define the *Synge's world function* [34] the bi-scalar

$$\sigma(x,x') = \frac{1}{2}(\lambda_1-\lambda_0)\int_{\lambda_0}^{\lambda_1} g_{ab}t^a t^b d\lambda\ , \tag{3.48}$$

with $\boldsymbol{t}$ the vector field tangent to the unique geodesic $\boldsymbol{l}$, which links x to x' in $\mathcal{O}$, with the integration to be taken along $\boldsymbol{l}$. Here, λ is an affine parameter of $\boldsymbol{l}$, such that $l(\lambda_0)=x$ and $l(\lambda_1)=x'$. If the geodesic is time-like, for instance, a good choice of the affine parameter is given by the proper time τ, for which $\sigma(x,x')=-\frac{1}{2}(\Delta\tau)^2$. Indeed, the world function (3.48) is half the squared geodesic distance between x and x'. Finally, definition (3.48) admits a well-posed definition in terms of the exponential map on $T_x\mathcal{M}$ [22].

The world function satisfies several properties [44]. Among these we recall

$$\begin{aligned}[\sigma] &= 0\ ,\\ \nabla_a\sigma\nabla^a\sigma &= 2\sigma\ .\end{aligned} \tag{3.49}$$

The half squared geodesic distance allows us to give a definition of physical admissible states on curved spacetimes. Indeed, we restrict our attention to those state whose singularities have a "physical meaning", by actually mimicking the divergent behaviour of ω on the Minkowski background.

Let ω_2 be the two-point correlation function of a state ω, and $t(x)$ be a time function on a time-orientable spacetime $(\mathcal{M},\boldsymbol{g})$. Let

$$\sigma_\epsilon(x,x') \doteq \sigma(x,x') + 2i\epsilon(t(x)-t(x')) + \epsilon^2\ .$$

We call ω a *quasi-free Hadamard state*, if

$$\omega_{abc'd'}(x,x') = \lim_{\epsilon\downarrow 0}\frac{1}{8\pi^2}\left(h^\epsilon_{abc'd'}(x,x') + w_{abc'd'}(x,x')\right)\ , \tag{3.50}$$

The bi-tensor $\boldsymbol{h}$ is called *Hadamard parametrix*, which is given by

$$h^{\epsilon}_{abc'd'}(x,x') \doteq \frac{u_{abc'd'}(x,x')}{\sigma_\epsilon(x,x')} + v_{abc'd'}(x,x')\log\left(\frac{\sigma_\epsilon(x,x')}{\lambda^2}\right) , \tag{3.51}$$

with λ an arbitrary length scale. The notion of quasi-free Hadamard state is the closest one to the idea of vacuum state of the flat theory. However, on curved spacetime the pure notion of vacuum cannot be fully recovered, since the ubiquitous presence of gravitational curvature gives rise to particle-production phenomena, which depend upon the choice of the reference frame [8].

On one hand, $w_{abcd}(x,x')$ is any arbitrary smooth symmetric bi-tensor, which is also regular for $x' \to x$. The freedom of $\boldsymbol{w}$ actually represents the arbitrariness of the notion of vacuum on curved spacetimes. Indeed, different $\boldsymbol{w}$ leads to different results of $\langle O\rangle_w$, which are related to different choices of states. This fact is deeply involved in the gravitational production of particles, which will be further discussed in section 3.6, by means of the existence of non-equivalent GNS representations of the same algebra of observables.

On the other hand, the Hadamard parametrix $h_{abc'd'}(x,x')$ completely describes the divergent behaviour of the state. Both $\boldsymbol{u}$ and $\boldsymbol{v}$ are smooth bi-tensors, hence, the singularities of the coinciding point limit of ω_2 are carried by the world function σ. Moreover, they are universal, not depending from the choice of the state, completely encoded by $\boldsymbol{w}$.

Since $\boldsymbol{w}$ is symmetric, the information regarding the dynamics of the quantum field $\hat{\gamma}$ is stored in $\boldsymbol{h}$, by means of the following prescription

$$iG_{abc'd'}(x,x') = \lim_{\epsilon\downarrow 0}(h^{\epsilon}_{abc'd'} - h^{-\epsilon}_{c'd'ab}(x',x))$$

3.5.2 Hadamard recursion relations

In the previous section we have introduced the notion of Hadamard quasi-free states, by means of the point-splitting prescription (3.50). However, further details on $\boldsymbol{u}$ and $\boldsymbol{v}$ can be exploited.

Let us consider the tensor differential operator associated with the Lichnerowicz equation of motion (2.25)

$$P^{ab}{}_{cd} = \delta^a_c\delta^b_d\Box - 2R^a{}_{cd}{}^b \ .$$

Since we are going to work with bi-tensors, we denote $P^{ab}{}_{cd}$, $P^{a'b'}{}_{c'd'}$ the tensor operators which respectively act on x and x'. As previously stressed by property (3.44), we shall work with those states which preserve the equations of motion. Indeed, given ω_2 the two-point function of ω, its distributional kernel satisfies the field equations

$$P^{ab}{}_{ef}\,\omega_{abc'd'} = 0, \quad P^{c'd'}{}_{e'f'}\,\omega_{abc'd'} = 0 \ .$$

If ω_2 satisfies the Hadamard condition (3.50), the equations of motion read

$$P^{ab}{}_{ef}\,(h^{\epsilon}_{abc'd'} + w_{abc'd'}) = 0 \ .$$

By substitution of (3.51), the equation of motion reads

$$P^{ab}{}_{ef}\left(\frac{u_{abc'd'}}{\sigma_\epsilon} + v_{abc'd'}\log\left(\frac{\sigma_\epsilon}{\lambda^2}\right)\right) = -P^{ab}_{ef}(w_{abc'd'}) \ . \tag{3.52}$$

Firstly, we observe that $\boldsymbol{h}$ and $\boldsymbol{w}$ does not *separately* satisfy the equation of motion. This fact will play a pivotal role in the following chapter, being responsible for the conservation anomaly of the stress-energy tensor.

Since $\boldsymbol{w}$ is smooth, then also the left-hand side of (3.52) is. By definition, both $\boldsymbol{u}$ and $\boldsymbol{v}$ are regular, thus, to achieve smoothness, we require that the singularities carried by $1/\sigma$ and $\log\sigma$ in (3.52) need to cancel each other out. We now the exploit the explicit computations behind this argument, which will allow us to derive a set of equations for $\boldsymbol{u}$ and $\boldsymbol{v}$.

As a first step, we consider the action of curved wave operator on the Hadamard parametrix $(\Box h)_{abc'd'}$. Without loss of generality, we assume $\lambda = 1$. The $\boldsymbol{v}$ contribution reads

$$\delta_a^e \delta_b^f \Box (v_{efc'd'} \log\sigma) = \nabla^\mu \left(\nabla_\mu v_{abc'd'} \log\sigma + v_{abc'd'} \nabla_\mu \log\sigma \right) \ .$$

Using the Leibniz rule, we obtain

$$\Box (v \log\sigma)_{abc'd'} = \Box v_{abc'd'} \log\sigma + 2\nabla_\mu v_{abc'd'} \nabla^\mu \log\sigma + v_{abc'd'} \Box \log\sigma \ . \tag{3.53}$$

Moreover, $\nabla^\mu \log\sigma = \frac{1}{\sigma}\nabla^\mu\sigma$. Taking the second derivative leads to

$$\Box \log\sigma = \tfrac{1}{\sigma}\Box\sigma - \tfrac{1}{\sigma^2}\nabla_\mu\sigma\nabla^\mu\sigma \ .$$

Finally, by substitution in (3.53), we exploit property (3.49), thus getting

$$\Box (v \log\sigma)_{abc'd'} = \Box v_{abc'd'} \log\sigma + 2\nabla_\mu v_{abc'd'} \nabla^\mu \log\sigma + v_{abc'd'} \tfrac{\Box\sigma}{\sigma} - v_{abc'd'} \tfrac{2}{\sigma} \ .$$

By considering the Riemann tensor term $R^e{}_{ab}{}^f v_{efc'd}$, we can extend this last result to $\boldsymbol{P}$, which reads

$$P (v \log\sigma)_{abc'd'} = P v_{abc'd'} \log\sigma + 2\nabla_\mu v_{abc'd'} \tfrac{\nabla^\mu\sigma}{\sigma} + v_{abc'd'} \tfrac{\Box\sigma}{\sigma} - v_{abc'd'} \tfrac{2}{\sigma} \ . \tag{3.54}$$

Similarly, we consider the $\boldsymbol{u}$ contribution, which gives

$$\Box \left(\frac{u}{\sigma}\right)_{abc'd'} = \nabla^\mu \left(\tfrac{1}{\sigma}\nabla_\mu u_{abc'd'} - \tfrac{1}{\sigma^2} u_{abc'd'} \nabla_\mu \sigma \right) \ .$$

Once again, using the Leibniz rule together with property (3.49) and adding the Riemann tensor term, we obtain

$$P \left(\tfrac{u}{\sigma}\right)_{abc'd'} = \tfrac{1}{\sigma} P u_{abc'd'} - \tfrac{2}{\sigma^2}\nabla_\mu u_{abc'd'} \nabla^\mu\sigma + u_{abc'd'} \tfrac{4}{\sigma^2} - \tfrac{\Box\sigma}{\sigma^2} u_{abc'd'} \ . \tag{3.55}$$

This equation does not show any term proportional to $\log\sigma$. Since the only contribution of this kind is carried by (3.54), to ensure (3.52) we need to require

$$(Pv)_{abc'd'} = 0 \ . \tag{3.56}$$

Since $\boldsymbol{v}$ is regular on the coinciding limit, we complete our discussion by considering its series expansion in σ

$$v_{abc'd'}(x, x') = \sum_{n=0}^{\infty} v^{(n)}_{abc'd'}(x, x') \sigma^n \ . \tag{3.57}$$

This expansion is not unique, unless we require $\boldsymbol{v}$ to satisfies the equation of motion (3.56). To show this, we substitute (3.57) and (3.56) in (3.54), thus getting

$$\begin{aligned} P(v \log \sigma)_{abc'd'} = 2 \sum_n \nabla_\mu v^{(n)}_{abc'd'} \sigma^{n-1} \nabla^\mu \sigma + 4 \sum_n n v^{(n)}_{abc'd'} \sigma^{n-1} + \\ + \sum_n v^{(n)}_{abc'd'} \sigma^{n-1} \Box \sigma - 2 \sum_n v^{(n)}_{abc'd'} \sigma^{n-1} \ . \end{aligned} \tag{3.58}$$

Adding this expression to (3.55), we ensure that $(Ph)_{abc'd'}$ is regular by equating order by order the divergent contributions of (3.55) and (3.58). For those terms which are singular by σ^{-2}, σ^{-1} and $\log \sigma$ we respectively get

$$2 \nabla_\mu \sigma \nabla^\mu u_{abc'd'} + (\Box - 4) u_{abc'd'} = 0 \ , \tag{3.59}$$

$$P u_{abc'd'} + \Box \sigma v^{(0)}_{abc'd'} - 2 v^{(0)}_{abc'd'} + 2 \nabla_\mu \sigma \nabla^\mu v^{(0)}_{abc'd'} = 0 \ , \tag{3.60}$$

$$P v_{abc'd'} = 0 \ . \tag{3.61}$$

These equations are called *Hadamard recursion relations* [22]. Moreover, a complete characterization of $v^{(n)}_{abc'd'}$ can be obtained by considering the series expansion (3.57) together with equation (3.61) [22, 23], giving the third recursion relation in the form

$$P v^{(n)}_{abc'd'} + 2(n+1) \nabla_\mu \sigma \nabla^\mu v^{(n+1)}_{abc'd'} + (n+1)(\Box \sigma + 2n) v^{(n+1)}_{abc'd'} = 0 \ . \tag{3.62}$$

The Hadamard recursion relations can be solved, provided some suitable initial conditions. By comparing the expression of ω_2 with the two-point correlation function of the flat theory, one requires that [23]

$$[u_{abc'd'}] = 1 \ .$$

Under this initial data, the solution to (3.59) can be written in terms of the Van Vleck-Morette determinant [22, 44, 1]

$$\Delta(x, x') \doteq \frac{\det(\nabla_{b'} \nabla_a \sigma(x, x'))}{\sqrt{-g}\sqrt{-g'}} \ , \tag{3.63}$$

with g' the metric determinant evaluated at x', such that [1]

$$u_{abc'd'}(x, x') = \Delta(x, x')^{1/2} g_{c'(a} g_{b)d'} \ .$$

Knowing $\boldsymbol{u}$, it is possible to derive the initial condition for (3.60), by considering its coinciding point limit which gives

$$[v^{(0)}_{abc'd'}] = \frac{1}{2} [P u_{abc'd'}] \ .$$

This procedure can be iterated to (3.62), finally leading to

$$[v^{(n+1)}_{abc'd'}] = \frac{1}{2(n+1)(n+2)} [P v^{(n)}_{abc'd'}] \ .$$

To summarize, in this section we have built a set of equations and initial condition which gives $\boldsymbol{u}$ and $\boldsymbol{v}$ as solutions, thus fixing the knowledge of the Hadamard parametrix $\boldsymbol{h}$, namely of the divergences of the two-point correlation function of linearized gravity. These results has been made possible by restricting to those states which satisfies the Hadamard condition (3.50), which shows a universal divergent behaviour, being independent from $\boldsymbol{w}$ and thus from the choice of the state, while still deeply related to the shape of the spacetime background. Finally, we point out that this procedure does not depend from the choice of the length parameter λ [23], whose role will be briefly discussed in section 3.5.4.

3.5.3 Taylor expansion around the coinciding point limit

Following the approach of [1], we can obtain further details and constraints on $\boldsymbol{v}^{(n)}$, which will get useful in the computation of the backreaction of the quantum field $\boldsymbol{\gamma}$. We start this section giving a useful definition.

Let us consider a vector field $\boldsymbol{V} \in T\mathcal{M}$. Let $T_x\mathcal{M}$ and $T_{x'}\mathcal{M}$ be two tangent spaces, i.e two fibers of $T\mathcal{M}$ respectively on x and x', with sections $\boldsymbol{V}(x)$ and $\boldsymbol{V}(x')$. We call *parallel propagator* [44], the map

$$p : T_x\mathcal{M} \to T_{x'}\mathcal{M} \ ,$$

which parallel-transports $\boldsymbol{V}(x)$ to $\boldsymbol{V}(x')$ along the unique geodesic that links x to x', such that

$$V^{a'}(x') = p^{a'}{}_a(x, x')V^a(x) \ . \tag{3.64}$$

Moreover, the coinciding point limit of $\boldsymbol{p}$ trivially reduces to $[p^{a'}{}_b] = \delta^a{}_b$.

As a first step, we consider further expansion of $\boldsymbol{v}^{(n)}$ as a Taylor series in σ, with x near x'. Actually, this reads

$$v^{(n)}_{abc'd'}(x,x') = p^c{}_{c'}p^d{}_{d'}[q^{(n)}_{abcd}(x) + \nabla_\mu q^{(n)}_{abcd}(x)\sigma^\mu + \tfrac{1}{2}\nabla_\nu\nabla_\mu q^{(n)}_{abcd}(x)\sigma^\mu\sigma^\nu + \ldots\ ] \ . \tag{3.65}$$

To notation extent, we define $z_{abcd}(x) \doteq q^{(1)}_{abcd}(x)$. By substitution of (3.65) into the Hadamard recursion relations, it can be shown that [1]

$$\begin{aligned} z_{ab}{}^{cd} =& \tfrac{1}{48}(g_a{}^c g_b{}^d + g_a{}^d g_b{}^c - g_{ab}g^{cd})(\tfrac{1}{30}R_{pqrs}R^{pqrs} - \tfrac{1}{30}R_{pq}R^{pq} + \tfrac{1}{12}R^2 + \tfrac{1}{5}\Box R) + \\ & - \tfrac{1}{24}(\Box + R)V_{ab}{}^{cd} + \tfrac{1}{8}V_{ab}{}^{pq}V_{pq}{}^{cd} + \tfrac{1}{48}\{(\Box + R)V_{abp}{}^p - 3V_{ab}{}^{pq}V_{pqr}{}^r\} + \\ & + \tfrac{1}{24}(R_{pq(a}{}^{(c}R_{b)}{}^{d)pq} - g_{(a}{}^{(c}R_{b)pqr}R^{d)pqr}) \ . \end{aligned}$$

For our purposes, the equation of motion (2.25) gives $V_{ab}{}^{cd} = -2R_a{}^{(c}{}_b{}^{d)}$. Tracing $\boldsymbol{z}$, we get

$$z^{acb}{}_c = \tfrac{47}{720}g^{ab}R^{pqrs}R_{pqrs} \ , \tag{3.66}$$

$$z^{abc}{}_c = -\tfrac{1}{720}g^{ab}R^{pqrs}R_{pqrs} \ . \tag{3.67}$$

In the following chapter, we will discuss the role of these relations in computation of the contribution of T_{ab} to the evaporation a spherically symmetric black hole.

3.5.4 Wick products and regularization

In this section we discuss the problem of the extension of the algebra $\mathcal{F}(\mathcal{M})$ to the coinciding point limit of those observables at least quadratic in the quantum field $\hat{\gamma}_{ab}(x)$. In section 3.3.1, we have already discussed that the construction of gauge-invariant squared observables brings some difficulty in the coinciding point limit. However, the introduction of ghost fields allowed us to simply write $\hat{\gamma}^2(\boldsymbol{f})$ without requiring anything on $\boldsymbol{f}$ and ensuring gauge-invariance when restricting to the cohomology of s.

During this section we take advantage of the notion of Hadamard quasi-free state (3.50), in order to regularize the expectation value of the simplest observable, e.g $\hat{\gamma}^2(\boldsymbol{f})$.

Let us consider a squared observable obtained by smearing the configuration $\hat{\gamma}_{ab}\hat{\gamma}_{c'd'}$. We compute its expectation value, which is given by the two-points correlation function

$$\langle \hat{\gamma}^2(f) \rangle_\omega = \int \omega_{abc'd'}(x,x') f^{abc'd'}(x) \delta^4(x-x') \sqrt{-g}\sqrt{-g'} d^4x d^4x' \ ,$$

at this level, we require ω to be a quasi-free state. We have already pointed out that the distributional character of $\hat{\gamma}_{ab}(x)$ brings a singularity in the coinciding point limit $[\omega_{abc'd'}]$. Indeed,

$$\langle [\hat{\gamma}_{ab}(x)\hat{\gamma}_{c'd'}(x')] \rangle_\omega \notin \mathcal{F}(\mathcal{M}) \ .$$

In quantum field theory on Minkowski spacetime, this problem is usually overcome with the introduction of the normal ordering prescription, by means of a divergences subtraction [46]. This procedure can be generalized to curved spacetimes following the so-called *Hadamard regularization*.

Let us consider an Hadamard quasi-free state ω, whose two-points correlation function is recalled

$$\omega_{abc'd'}(x,x') = \lim_{\epsilon \downarrow 0} \frac{1}{8\pi^2} \left(h^{\epsilon}_{abc'd'}(x,x') + w_{abc'd'}(x,x') \right) \ .$$

In section 3.5.1 we have stated that the divergent behaviour of $\omega_{abc'd'}(x,x')$ is completely described by the Hadamard parametrix $\boldsymbol{h}$. To regularize the expectation value of $\hat{\gamma}^2$, we subtract the $\boldsymbol{h}$ term, by means of the point-splitting procedure [22, 31]

$$:\hat{\gamma}_{ab}(x)\hat{\gamma}_{c'd'}(x'): \doteq \hat{\gamma}_{ab}(x)\hat{\gamma}_{c'd'}(x') - \tfrac{1}{8\pi^2} h_{abc'd'}(x,x')\hat{\mathbb{I}} - g_{ab}g_{c'd'}H(x,x')\hat{\mathbb{I}} \ , \tag{3.68}$$

with $H(x,x')$ any bi-scalar, regular on the coinciding point limit. This procedure actually makes sense, being independent from the choice of the Hadamard state, for which, we recall, the divergent behaviour encoded by the parametrix $\boldsymbol{h}$ is universal. Hence, we start from the expectation value

$$\langle :\hat{\gamma}_{ab}(x)\hat{\gamma}_{c'd'}(x'): \rangle_\omega = \frac{1}{8\pi^2} w_{abc'd'}(x,x') - g_{ab}g_{c'd'}H(x,x') \ ,$$

to get a regularized Wick monomial

$$\langle :\hat{\gamma}_{ab}(x)\hat{\gamma}_{cd}(x): \rangle_\omega = \frac{1}{8\pi^2}[w_{abcd}] - g_{ab}g_{cd}[H] \ , \tag{3.69}$$

which, due to the smoothness of $\boldsymbol{w}$, is finally well-behaved on the coinciding point limit $x' \to x$, which gives[5]

$$\langle : \hat{\gamma}^2(\boldsymbol{f}) : \rangle_\omega = \frac{1}{8\pi^2} \int [w_{abcd}](x) f^{abcd}(x) \sqrt{-g} d^4x \ .$$

The point-splitting procedure can be iterated to higher orders terms, which allows the definition of the n-th *Wick monomials* from

$$: \hat{\gamma}(\boldsymbol{f_1}) \cdots \hat{\gamma}(\boldsymbol{f_n}) : \doteq \int : \hat{\gamma}_{a_1 b_1}(x_1) \cdots \hat{\gamma}_{a_n b_n}(x_n) : f^{a_1 b_1}(x_1) \cdots f^{a_n b_n}(x_n) dvol_g^n \ ,$$

where the normal-ordered kernel can be obtained by means of the Wick theorem [22, 31] and

$$dvolg_g^n = (-g(x_1))^{1/2} \cdots (-g(x_n))^{1/2} d^4x_1 \cdots d^4x_n \ .$$

We should note that, following the BRST approach, there is no need to require additional conditions on the test tensors. In contrast to the approach of section 3.3.1, $: \hat{\gamma}^2(\boldsymbol{f}) :$ is not gauge-invariant. This property is achieved by considering the observables as those object which are s-invariant, where the gauge-dependent contribution are compensated by ghost fields.

From this discussion we can observe how the singularities subtraction is made possible by the property of Hadamard states to have a universal divergent behaviour. The possibility to regularize the Wick monomials guarantees that the algebra of observables is well-defined even in the coinciding point limit, allowing for the inclusion of $: \hat{\gamma}^n(\boldsymbol{f}) :$ in the algebra of fields $\mathcal{F}(\mathcal{M})$. Moreover, when considering the flat limit, the Hadamard regularization procedure correctly reduces to the the normal ordering one, where putting all the annihilation operators to the right of the creation ones actually leads to the subtraction of the contribution of the Minkowski vacuum.

Before ending this section we point out that the point-splitting prescription (3.68) is not unique at all. On one hand, it is always possible to consider any arbitrary but regular bi-scalar H, which does not modify the divergences subtraction of $\boldsymbol{h}$, while contributing to the expectation value (3.69). On the other hand, since $\boldsymbol{v}$ is smooth it is always possible to choice a length parameter $\lambda \neq 1$ which contributes to the Hadamard parametrix (3.51), with no need to be subtracted in the point-splitting prescription (3.68), being regular for $x' \to x$. Finally, these considerations can be resumed by stating that there is at least a two-fold regularization freedom, which gives the following contributions to the expectation value of the Wick monomial

$$\langle : \hat{\gamma}_{ab}(x) \hat{\gamma}_{cd}(x) : \rangle_\omega = \frac{1}{8\pi^2} [w_{abcd}] - 2\log(\lambda)[v_{abcd}] - g_{ab} g_{cd} [H] \ .$$

This freedom will actually play a pivotal role in the following chapter, where it will be exploited to get regularized and covariantly conserved contribution of the stress-energy tensor, while producing an anomaly in its trace.

[5]As discussed in section 3.3.1, the coinciding point limit of observables beyond linear order require the use of higher rank test tensors, which can describe the freedom in contracting the configuration fields

3.5.5 Hadamard states and ghost fields

In this section we briefly discuss the property of Hadamard states for ghost fields. Previously we have considered two vector fields $\boldsymbol{c}$, $\bar{\boldsymbol{c}}$, which on a vacuum spacetime (1.6) satisfy

$$\Box c^a = 0, \quad \Box \bar{c}^a = 0 \ .$$

The quantization has be achieved by means of the algebraic approach, already discussed in section 3.4, giving

$$\begin{aligned} \{\hat{c}_a(x), \hat{\bar{c}}_b(x')\} &= iE_{ab}(x,x')\hat{\mathbb{I}} \ , \\ \{\hat{c}_a(x), \hat{c}_b(x')\} &= \{\hat{\bar{c}}_a(x), \hat{\bar{c}}_b(x')\} = 0 \ , \end{aligned}$$

thus giving rise to the unital $*$-algebra of ghost fields, which we shall denote $\mathcal{C}(\mathcal{M})$ and which is generated by $\hat{c}(h)$, $\hat{\bar{c}}(h)$ and $\hat{\mathbb{I}}$.

Previously, we have defined the quantum state ω as functional on the algebra of fields $\mathcal{F}$ (3.42), which contains both the contribution coming from ghosts and the perturbation field $\hat{\gamma}$. We are now interested in discussing the notion of Hadamard state on $\mathcal{C}(\mathcal{M})$. Indeed, we consider a quasi-free state ω, whose two-point correlation function is given by

$$\omega_{ab'}(x,x') = \langle \hat{c}_a(x)\hat{c}_b(x')\rangle_\omega \ .$$

We call ω a quasi-free Hadamard state if its two-point function satisfies

$$\omega_{ab'}(x,x') = \lim_{\epsilon\downarrow 0} \frac{i}{8\pi^2}\left(\tilde{h}^\epsilon_{ab'}(x,x') + \tilde{w}_{ab'}(x,x')\right) \ , \tag{3.70}$$

with the Hadamard parametrix $\tilde{\boldsymbol{h}}$ given by

$$\tilde{h}^\epsilon_{ab'}(x,x') \doteq \frac{\tilde{u}_{ab'}(x,x')}{\sigma_\epsilon(x,x')} + \tilde{v}_{ab'}(x,x')\log\left(\frac{\sigma_\epsilon(x,x')}{\lambda^2}\right) \ . \tag{3.71}$$

Once again, the parametrix encodes the divergent behaviour of ω_2, while $\tilde{\boldsymbol{w}}$ is a smooth symmetric bi-tensor, which describes the freedom in the choice of the state. Using the notion of Hadamard state, we can repeat the construction of section 3.5.4, enlarging the algebra $\mathcal{C}(\mathcal{M})$ by means of the Hadamard regularization prescription. Indeed, on $\mathcal{C}(\mathcal{M})$ the point-splitting prescription reads

$$:\hat{c}_a(x)\hat{c}_{b'}(x'): \doteq \hat{c}_a(x)\hat{c}_{b'}(x') - \tfrac{i}{8\pi^2}h_{ab'}(x,x')\hat{\mathbb{I}} \ ,$$

leading to the following regularized correlation function

$$\langle :\hat{c}_a(x)\hat{c}_b(x):\rangle_\omega = \frac{i}{8\pi^2}[w_{ab}] \ ,$$

which is now smooth on the coinciding point limit.

Again, we consider the following series expansion around the coinciding point limit

$$\tilde{v}_{ab'}(x,x') = \sum_{n=0}^{\infty} \tilde{v}^{(n)}_{ab'}(x,x')\sigma^n \ .$$

By means of the same procedure of section 3.5.2, we impose the equation of motion $\Box(\tilde{\boldsymbol{h}}+\tilde{\boldsymbol{w}})=0$, isolating the singularities order by order and thus obtaining the Hadamard recursion relations for ghost fields

$$\begin{aligned}&2\nabla_\mu\sigma\nabla^\mu\tilde{u}_{ab'}+(\Box-4)\tilde{u}_{ab'}=0\ ,\\&\Box\tilde{u}_{ab'}+\Box\sigma\tilde{v}^{(0)}_{ab'}-2\tilde{v}^{(0)}_{ab'}+2\nabla_\mu\sigma\nabla^\mu\tilde{v}^{(0)}_{ab'}=0\ ,\\&\Box\tilde{v}^{(n)}_{ab'}+2(n+1)\nabla_\mu\sigma\nabla^\mu\tilde{v}^{(n+1)}_{ab'}+(n+1)(\Box\sigma+2n)\tilde{v}^{(n+1)}_{ab'}=0\ .\end{aligned}$$

Going a little further, we consider of Taylor expansion of $\tilde{\boldsymbol{v}}^{(\boldsymbol{n})}$, giving

$$\tilde{v}^{(n)}_{ab'}(x,x')=p^c{}_{c'}p^d{}_{d'}[\tilde{q}^{(n)}_{ab}(x)+\nabla_\mu\tilde{q}^{(n)}_{ab}(x)\sigma^\mu+\tfrac{1}{2}\nabla_\nu\nabla_\mu\tilde{q}^{(n)}_{abcd}(x)\sigma^\mu\sigma^\nu+...]\ ,\tag{3.72}$$

with $\boldsymbol{p}$ the parallel propagator, as defined by (3.64). Denoting $\tilde{z}_{ab}(x)\doteq\tilde{q}^{(1)}_{ab}(x)$ and by substitution of (3.72) into the Hadamard recursion relations, it follows that on vacuum spacetimes [1]

$$\tilde{z}^{ab}=-\tfrac{11}{2880}g^{ab}R^{pqrs}R_{pqrs}\ .\tag{3.73}$$

This result, together with (3.66) and (3.67), will play an important role in understanding the trace anomaly of linearized quantum gravity and its role on the evaporation of black holes.

3.6 Recovering Fock space. The GNS construction and particle creation

In this chapter we have described the algebraic approach to quantum field theory, specializing it to case of linearized quantum gravity. Indeed, the powerful techniques of AQFT have provided a toolkit to achieve quantization on curved spacetimes, via a canonical covariant fashion. Indeed, it has been possible to describe the entire theory, from the algebra of observables to the computation of expectation values, without referring to any particular representation on some Hilbert space.

We conclude this chapter by explaining how the algebraic approach can be successfully represented on Hilbert spaces, thus recovering the usual formalism of quantum mechanics. Except for few details, the following construction holds for different physical models and quantum fields.

We begin this discussion by giving the following definition. We start from $\mathcal{A}(\mathcal{M})$, the unital $*$-algebra of perturbation fields[6], on which we consider a positive state ω. Let $\mathcal{H}$ be an Hilbert space, $\mathcal{D}$ a dense subspace of $\mathcal{H}$ and $\mathcal{L}(\mathcal{D})$ the space of linear operator on $\mathcal{D}$. We call *$*$-representation* of $\mathcal{A}(\mathcal{M})$, any linear, product-and-unit preserving map

$$\pi:\mathcal{A}(\mathcal{M})\to\mathcal{L}(\mathcal{D})\ ,$$

such that for any $O\in\mathcal{F}(\mathcal{M})$

$$\pi(O)^\dagger|_{\mathcal{D}}=\pi(O^*)\ .$$

[6]At the moment we are not accounting for the contribution of ghost fields. The reason behind this choice will become clear at the end of this section

Moreover, we say that a vector Ψ *spans* $\mathcal{D}$, if

$$\pi(\mathcal{A})\Psi = \mathcal{D} \ .$$

From the algebraic point of view, the quantum field $\hat{\gamma}$ has been defined in an abstract fashion as a generator of the $*$-algebra of fields. The choice of a $*-$representation π, allows us to actually map $\hat{\gamma}$ into the space of operator valued distributions, recovering the common interpretation of quantum field. This idea can be mathematically implemented through the Gelfand–Naimark–Segal (GNS) construction [31].

Theorem 2. *Let $\mathcal{A}$ be a unital $*$-algebra and ω a positive state. By means of the previous definitions, there exists a quadruple $(\mathcal{H}, \mathcal{D}, \pi, \Psi)$, respectively given by an Hilbert space $\mathcal{H}$, a dense subspace $\mathcal{D}$, a $*$-representation π and a vector Ψ, such that, given $O \in \mathcal{A}$*

$$\omega(O) = \langle \Psi | \pi(O) | \Psi \rangle \ .$$

Moreover, if another quadruple $(\mathcal{H}', \mathcal{D}', \pi', \Psi')$ satisfies,

$$\omega(O) = \langle \Psi' | \pi'(O) | \Psi' \rangle \ .$$

then there exist an isometry $U : \mathcal{H} \to \mathcal{H}'$, such that for any $O \in \mathcal{A}$

$$\begin{aligned} U\Psi &= \Psi' \ , \\ U\mathcal{D} &= \mathcal{D}' \ , \\ U\pi(O)U^{-1} &= \pi'(O) \ . \end{aligned}$$

The GNS theorem guarantees that from the abstract, but powerful, algebraic point of view, it is always possible to recover a representation of $\mathcal{A}$ on some Hilbert space, which preserves the notion of expectation value of observables. A stronger version of this theorem holds for quantum mechanics, in which the quadruple reduces to the GNS triple $(\mathcal{H}, \pi, \Psi)$, where Ψ is a *cyclic* vector, being enough to span the entire Hilbert space

$$\pi(\mathcal{A})\Psi = \mathcal{H}$$

However, this is not the case of quantum field theory, where the infinitely many degrees of freedom allow only for a map of $\mathcal{A}$ into a subspace of $\mathcal{D}$.

As previously mentioned, theorem 2 cannot be trivially applied to $\mathcal{F}(M)$, due to the presence of ghost fields, which leads to states that fails the positivity requirement of ω. However, the GNS-construction can be extended, by considering an indefinite scalar product space, i.e Krein space rather than an Hilbert space [31, 26]

If ω is a (pure) quasi-free state, the GNS quadruple builds an irreducible representation of $\mathcal{A}$ of the one-particle structure on the Fock space, with $|\Psi\rangle$ as the vacuum state of the theory. In some sense, this justifies the description of quasi-free states in terms of those functionals on $\mathcal{A}$ which allow for a (at least asymptotic) recovery of the notion of particle [22].

As discussed at the beginning of this chapter, a curved spacetimes admit different non-equivalent representations of the same algebra, which both satisfies the GNS construction, while giving

$$\langle \Psi | \pi(O) | \Psi \rangle \neq \langle \Psi' | \pi'(O) | \Psi' \rangle \ .$$

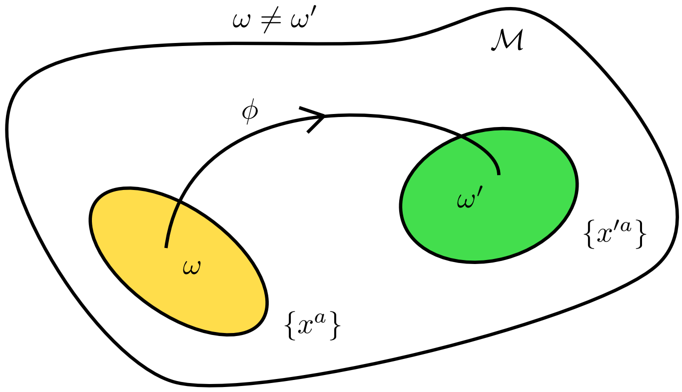


Figure 3.1: Multiple states from diffeomorphisms invariance

Indeed, let us consider the Fock representation of $\hat{h} = \pi(\hat{\gamma})$, in terms of creation and annihilation operators, with vacuum state $|\Omega\rangle$, such that

$$\langle\Omega|h_{ab}|\Omega\rangle = 0 \ .$$

However, we may always find another non-equivalent representation of $\mathcal{A}$ such that, if $U : \mathcal{H} \to \widetilde{\mathcal{H}}$,

$$\langle\tilde{\Omega}|\tilde{h}_{ab}|\tilde{\Omega}\rangle \neq 0,$$

with

$$U h_{ab} U^{-1} \neq \tilde{h}_{ab} \ .$$

Hence, the notion of vacuum is ill-defined, since it depends from the choice of the representation, related on its turn to the choice of the quantum state ω. This fact bring non-negligible physical consequences. For instance, due to the curvature of spacetime, a free falling observers will measure some effects related to the presence of particles (of any kind), which has been gravitationally produced.

This discussion justifies, a posteriori, how the algebraic approach reveals to be more adapt when dealing with quantum fields on curved spacetimes. Contrary to the standard approach, splitting the construction of states and of representations from the rest of the discussion, has made possible to give a covariant quantization procedure, which holds for any choice of the spacetime background. On the other hand, we have shown that different choices of backgrounds and states leads to non-negligible consequences in the computation of the expectation values of gauge-invariant observables. However, by the choice of an Hadamard state

$$\omega_{abc'd'}(x, x') = \lim_{\epsilon \downarrow 0} \frac{1}{8\pi^2} \left(h^{\epsilon}_{abc'd'}(x, x') + w_{abc'd'}(x, x') \right) \ ,$$

it has been possible to account for these ambiguities, in some sense related to the gravitational distortion of vacuum, in terms of a freedom in the choice of the smooth symmetric tensor $\boldsymbol{w}$, and thus of the GNS representation on some Hilbert space.

Part III

From quantum gravitational radiation to black hole evaporation

Chapter 4

Gravitational evaporation of a Schwarzschild black hole

4.1 Introduction

In this chapter we discuss the role of quantum gravitational radiation in black evaporation, by studying the effect of a metric perturbation in the Raychaudhuri's equation. From section 1.5 we recall that, for a radial null outgoing geodesics congruence, the perturbed Raychaudhuri's equation reads

$$\left.\frac{d\theta}{d\lambda}\right|_{\mathcal{H}^+} = -\varepsilon^2 \hat{\sigma}^{(1)}_{ab} \hat{\sigma}^{ab}_{(1)} - \varepsilon^2 R^{(2)}_{cd} k^c k^d + \mathcal{O}(\varepsilon^3) \; . \tag{4.1}$$

The computation of these contributions requires two different interpretations on the role of the perturbation field γ_{ab}

1. From a geometrical point of view, we can evaluate how γ_{ab} contributes to the squared shear tensor $\hat{\sigma}_{ab}\hat{\sigma}^{ab}$, both explicitly through the metric and by means of a correction to the geodesics tangent vector field.

2. By means of more field-oriented point of view, we can study the second order Ricci term, by actually relating it to a semiclassical contribution produced by the stress-energy tensor of $\boldsymbol{\gamma}$.

In both cases, the presence of quantum gravitational radiation is hidden in $k^c k^d R^{(2)}_{cd}$ and $\hat{\sigma}^{(1)}_{ab} \hat{\sigma}^{ab}_{(1)}$, which will be considered as observables of our theory and computed by means of a semiclassical approach, under the choice of a suitable quantum state.

In the previous chapters we have studied quantum gravitational radiation from the point of view of algebraic quantum field theory on curved spacetime. Indeed, our discussion has begun from the linearization of the complete metric

$$\tilde{g}_{ab} = g_{ab} + \varepsilon \gamma_{ab} + \mathcal{O}(\varepsilon^2) \; ,$$

around a fixed classical background spacetime $(\mathcal{M}, \boldsymbol{g})$. The purpose of semiclassical gravity is to take a step forward towards a theory of quantum gravity.

We begin our discussion by assuming that the spacetime background may actually change under the effect of finite contribution produced by the backreaction of a quantum field like $\hat{\gamma}$ [22]. In the following sections we shall investigate the role of the stress-energy tensor of quantized gravitational radiation, by applying the Hadamard point-splitting procedure, already introduced in section 3.5.4, and focusing on its physical consequences.

Let us consider a quantum state ω. The computation of the backreaction of a quantum matter field can be achieved by means of the *semiclassical Einstein equation* [22, 47, 33]

$$\widetilde{G}_{ab} = 8\pi \langle T_{ab} \rangle_\omega \ , \tag{4.2}$$

with $G = 1$ and $\boldsymbol{T}$ the stress-energy tensor of the quantum theory. Actually, using (4.2) together with (1.5) and recalling that, on the background spacetime the congruence satisfies $k_a k^a = 0$, we get the *semiclassical Raychaudhuri's equation*, which for linearized gravity actually reads

$$\left.\frac{d\theta}{d\lambda}\right|_{\mathcal{H}^+} = -\varepsilon^2 \langle \hat{\sigma}^{(1)}_{ab} \hat{\sigma}^{ab}_{(1)} \rangle_\omega - \varepsilon^2 8\pi \langle T_{ab} \rangle_\omega k^a k^b + \mathcal{O}(\varepsilon^3) \ . \tag{4.3}$$

Since T_{ab} is quadratic in $\hat{\gamma}_{ab}$, in the following section we shall recall its renormalization [1], which will lead to a quantum conserved prescription for $\langle T_{ab} \rangle_\omega$ [22], to be used in (4.3).

The computation of the explicit expression of the squared shear tensor $\hat{\sigma}^{(1)}_{ab} \hat{\sigma}^{ab}_{(1)}$ is considered in appendix B. As we shall see in the following section for T_{ab}, we should continue this analysis by studying the renormalization of $\langle \hat{\sigma}^{(1)}_{ab} \hat{\sigma}^{ab}_{(1)} \rangle_\omega$, thus obtaining its contribution to the backreaction. However, the expression of the squared shear tensor found in B is not so easy to treat. For this reason, we prefer to follow the second approach explained in 2, by interpreting $\hat{\gamma}_{ab}$ as an external quantum field which propagates on $(\mathcal{M}, \boldsymbol{g})$, focusing on the contribution to the backreaction given by its stress-energy tensor. A posteriori, due to some similarities between the contribution of $\hat{\sigma}^{ab} \hat{\sigma}_{ab}$ and T_{ab} (e.g the term $k^i k^f \nabla_f \gamma^{dk} \nabla_i \gamma_{dk}$) and under some physical requirement on the state ω, one should expect approaches 1 and 2 to give compatible results, such that

$$\langle : \hat{\sigma}^{ab} \hat{\sigma}^{(ren)}_{ab} : \rangle_\omega |_{\mathcal{H}^+} \propto \langle : T^{(ren)}_{ab} : \rangle_\omega k^a k^b |_{\mathcal{H}^+} \ .$$

At the end of this chapter we shall also consider the evaporation induced by a massless scalar field ϕ backreacting on the spacetime background, which can be computed in a similar way.

4.2 Regularization of the stress-energy tensor

We start by briefly resuming the result of [1, 13, 17], adopting them to the framework of algebraic quantum field theory on curved spacetime [22, 47, 31, 7].

We begin our discussion from the second order expansion of the Einstein-Hilbert action, which gives the action of linearized gravity S_{lg} (2.16)

$$S_{lg} = \frac{1}{32\pi G} \int [\tfrac{1}{2} \gamma^{ab} \Box \gamma_{ab} - \tfrac{1}{4} \gamma \Box \gamma + (\nabla^a \gamma_{ab} - \tfrac{1}{2} \nabla_b \gamma)^2 + \gamma^{ab} R_{acbd} \gamma^{cd} + \\ + \gamma^a{}_b R^{bc} \gamma_{ac} - \gamma \gamma^{ab} R_{ab} - \tfrac{1}{2} R \gamma_{ab} \gamma^{ab} + \tfrac{1}{4} R \gamma^2] \sqrt{-g} d^4 x \ .$$

We follow the approach of [1], which is near to the one of [17, 16], already resumed in section 3.4. Indeed, the quantization of the theory is performed by means of the Fadeev-Popov procedure, adding a de Donder gauge-fixing term S_{gf} [1]

$$S_{gf} = -\frac{1}{32\pi G}\int \frac{1}{2\alpha}[\nabla^a\gamma_{ab} - k\nabla_b\gamma^a{}_a][\nabla_c\gamma^{cb} - k\nabla^b\gamma^c{}_c]\sqrt{-g}d^4x \ ,$$

compensated by the ghost fields $\boldsymbol{c}$ and $\bar{\boldsymbol{c}}$, whose action is given by [1]

$$S_{gh} = -\frac{1}{32\pi G}\int \bar{c}_a[g^{ab}\Box + R^{ab}]c_b\sqrt{-g}d^4x \ .$$

Here, we have modified the convention of section 3.5, following the one of [1], by removing the i factor from both the action (the stress-energy tensor) and the two-point correlation function, which would be anyway cancelled by the $-i$ factor within the definition of the matrix $\omega(F)$ (3.46).

We define the stress-energy tensor as[1]

$$T^{ab} \doteq -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{ab}} \ , \tag{4.4}$$

with $S \doteq S_{lg} + S_{gf} + S_{gh}$. According to the previous definition (4.4), the different terms of the gauge-invariant action leads to three different contribution to T^{ab}. For a vacuum spacetime the gravitational contribution gives [1]

$$\begin{aligned}
T^{\mu\nu}_{lg} = \frac{1}{32\pi G}\big[& + \nabla^\mu\gamma^{ab}\nabla^\nu\gamma_{ab} - 4\gamma^{b(\nu}\nabla^{\mu)}\nabla^a\gamma_{ab} + 4\gamma^{b(\nu}\nabla^{\mu)}\nabla_b\gamma - 4\nabla^b\nabla^{(\nu}\gamma^{\mu)a}\gamma_{ab} + \\
& - 4\nabla^{(\nu}\gamma^{\mu)a}\nabla^b\gamma_{ab} - 2\nabla_b\nabla_a\gamma^{a(\mu}\gamma^{\nu)b} - 2\nabla_a\nabla_b\gamma^{a(\mu}\gamma^{\nu)b} + \\
& - 2\nabla_b\gamma^{a(\mu}\nabla_a\gamma^{\nu)b} + 2\nabla_b\gamma\nabla^{(\nu}\gamma^{\mu)b} + 2\nabla_b\nabla_a\gamma^{ab}\gamma^{\mu\nu} + 2\nabla_b\nabla_a\gamma^{\mu\nu}\gamma^{ab} + \\
& + 2\nabla_a\gamma^{\mu\nu}\nabla_b\gamma^{ab} + 2\nabla_b\gamma^\mu{}_a\nabla^b\gamma^{\nu a} - \nabla_a\gamma\nabla^a\gamma^{\mu\nu} + 2\gamma_{ab}\nabla^{(\nu}\nabla^{\mu)}\gamma^{ab} + \\
& - \gamma\nabla^\nu\nabla^\mu\gamma + 2\gamma\nabla_b\nabla^{(\nu}\gamma^{\mu)b} + 2\gamma^{ab}\gamma^{c(\mu}R^{\nu)}{}_{abc} - \gamma\Box\gamma^{\mu\nu} + \\
& - g^{\mu\nu}(+2\gamma^{ab}\Box\gamma_{ab} - \gamma\Box\gamma + \tfrac{3}{2}\nabla^c\gamma^{ab}\nabla_c\gamma_{ab} - \nabla_a\gamma^{ab}\nabla^c\gamma_{bc} + \\
& - 3\nabla^c\nabla^a\gamma_a{}^b\gamma_{bc} + \gamma^{ab}R_{acbd}\gamma^{cd} - \gamma^{ab}\nabla_c\nabla_a\gamma^c{}_b - \tfrac{1}{2}\nabla^a\gamma\nabla_a\gamma + \\
& + 2\nabla^a\gamma\nabla^b\gamma_{ab} + \nabla^a\nabla^b\gamma\gamma_{ab} + \gamma\nabla_a\nabla_b\gamma^{ab} - \nabla^c\gamma^{ab}\nabla_a\gamma_{bc} \ )\big] \ ,
\end{aligned}$$

while the de Donder gauge-fixing contribution is

$$\begin{aligned}
T^{\mu\nu}_{gf} = \frac{1}{32\pi G}\big[& + 4\gamma^{b(\mu}\nabla^{\nu)}\nabla^a\gamma_{ab} - 2\gamma^{b(\mu}\nabla^{\nu)}\nabla_b\gamma + 2\nabla_a\gamma^{\mu\nu}\nabla_b\gamma^{ab} + \\
& - \nabla_a\gamma\nabla^a\gamma^{\mu\nu} - 2\nabla_a\gamma^{\mu a}\nabla_b\gamma^{\nu b} + 2\nabla_b\gamma^{b(\mu}\nabla^{\nu)}\gamma - \tfrac{1}{2}\nabla^\mu\gamma\nabla^\nu\gamma + \\
& - g^{\mu\nu}(+\nabla_a\gamma^{ab}\nabla^c\gamma_{bc} + 2\nabla^c\nabla^a\gamma_a{}^b\gamma_{bc} - \tfrac{1}{4}\nabla^a\gamma\nabla_a\gamma - \nabla^a\nabla^b\gamma\gamma_{ab} \ )\big] \ .
\end{aligned}$$

On the other hand, the ghost term reads

$$\begin{aligned}
T^{\mu\nu}_{gh} = -\frac{1}{32\pi G}\big[& + 2\nabla_b\bar{c}^b\nabla^{(\mu}c^{\nu)} + 2\nabla^{(\mu}\bar{c}^{\nu)}\nabla_b c^b + 2\bar{c}_a\nabla^a\nabla^{(\mu}c^{\nu)} + \\
& + 2c_a\nabla^a\nabla^{(\mu}\bar{c}^{\nu)} + 2\nabla^{(\mu}\bar{c}_a\nabla^{\nu)}c^a - 2\nabla^a\bar{c}^{(\mu}\nabla_a c^{\nu)} + \\
& - g^{\mu\nu}(+2\nabla_a\bar{c}_b\nabla^{(a}c^{b)} + \nabla_{(a}\nabla_{b)}\bar{c}^a c^b + \bar{c}^b\nabla_{(a}\nabla_{b)}c^a + \\
& - \nabla_a\bar{c}^a\nabla_b c^b)\big] \ .
\end{aligned}$$

[1] Here, we are using the opposite notation with respect to [1], in order to avoid a minus sign in the Einstein equation.

From now on we denote

$$\begin{aligned} T^{ab}_{dD} &\doteq T^{ab}_{lg} + T^{ab}_{gf} \ , \\ T^{ab} &\doteq T^{ab}_{lg} + T^{ab}_{gf} + T^{ab}_{gh} \ , \end{aligned} \tag{4.5}$$

respectively the gauge-fixed and the total stress-energy tensor. As we have pointed out several times, T_{ab}, currently defined by the functional derivative of S, is quadratic in the perturbation field γ_{ab} and its derivatives, hence a direct computation of its expectation would be divergent on the coinciding point limit. To prevent this, we shall consider the normal ordering prescription, already discussed in section 3.5.4, applying it to T_{ab}. As we shall see in the next section, this procedure will lead to a conservation anomaly in $\langle : T_{ab} : \rangle$, which will need to be removed via a renormalization, thus being finally able to write the semiclassical Einstein equation.

Let us associate a bi-tensor to any contribution of T_{ab}, by simply assuming that all the products of fields are taken at two different points of the spacetime x and x', such that

$$T^{ab'}(x,x') \doteq T^{ab'}_{lg}(x,x') + T^{ab'}_{gf}(x,x') + T^{ab'}_{gh}(x,x') \ .$$

To reduce our notation, we can write T^{ab} in terms of the action of a bi-differential operator, distinguishing between the gauge-fixed and ghost contributions as

$$T^{ab'}_{dD} \doteq D^{ab'cde'f'}_{dD}(\gamma_{cd}, \gamma_{e'f'}) \ , \tag{4.6}$$

$$T^{ab'}_{gh} \doteq D^{ab'cd'}_{gh}(c^c, c^{d'}) \ . \tag{4.7}$$

Finally, if we consider a state ω, the distributional character of both γ_{ab} and c^a leads to

$$[\langle T_{ab}\rangle_\omega](x) \doteq \lim_{x'\to x} \langle T_{ab'}(x,x')\rangle_\omega \to \infty \ .$$

As already done in section 3.5.4, we solve this critical issue by employing the normal ordering procedure.

Let us consider an Hadamard quasi free state ω, which acts on both the ghost and perturbation quantum fields by means of (3.50) and (3.70), which we recall being

$$\omega_{abc'd'}(x,x') = \lim_{\epsilon\downarrow 0} \frac{1}{8\pi^2}\left(h^\epsilon_{abc'd'}(x,x') + w_{abc'd'}(x,x')\right) \ , \tag{4.8}$$

$$\omega_{ab'}(x,x') = \lim_{\epsilon\downarrow 0} \frac{1}{8\pi^2}\left(\tilde{h}^\epsilon_{ab'}(x,x') + \tilde{w}_{ab'}(x,x')\right) \ , \tag{4.9}$$

where $h^\epsilon_{abc'd'}$ and $\tilde{h}^\epsilon_{ab'}$ are respectively the Hadamard parametrices associated to γ_{ab} and c^a, given by (3.51) and (3.71).

We can apply the point splitting procedure of section 3.5.4 to (4.6) and (4.7). Indeed, by subtraction of the divergent terms encoded by the Hadamard parametrices of (4.8) and (4.9), we get the following results

$$\begin{aligned} \langle : T^{ab}_{dD} : \rangle_\omega &= \frac{1}{8\pi^2}[D^{ab'cde'f'}_{dD} w_{cde'f'}](x) \ , \\ \langle : T^{ab}_{gh} : \rangle_\omega &= \frac{1}{8\pi^2}[D^{ab'cd'}_{gh} \bar{w}_{cd'}](x) \ , \end{aligned}$$

which stand as a generalization of the normal ordering prescription to curved spacetimes. Since both $w_{cde'f'}$ and $w_{cd'}$ are smooth on the coinciding point limit, the expectation value of the regularized (i.e normal ordered) stress-energy tensor is finite, finally giving

$$\langle : T^{ab} : \rangle_\omega = \frac{1}{8\pi^2} \left([D^{ab'cde'f'}_{dD} w_{cde'f'}] + [D^{ab'cd'}_{gh} \bar{w}_{cd'}] \right) \ . \tag{4.10}$$

As previously discussed in section 3.5.4, this prescription is not unique, due to the freedom encoded in the choice of length parameter within the parametrices $h^\epsilon_{abc'd'}$ and $\tilde{h}^\epsilon_{ab'}$, together with the possibility to always redefine (4.10) by the subtraction of a tensor contribution regular on the coinciding point limit.

4.3 Renormalization of the stress-energy tensor

In the previous section we have obtained an expectation value of the stress-energy tensor which is well-behaved (4.10). Indeed, by the choice of an Hadamard state ω, we can actually compute $\langle : T^{ab} : \rangle_\omega$. Anyway, as we shall discuss, we are not ready to write down the semiclassical Einstein equation (4.2).

The previously discussed Hadamard regularization procedure brings another critical issue: the regularized stress-energy tensor constructed by a simple subtraction of the divergent terms (4.10) is not covariantly conserved, being $\nabla^a \langle : T_{ab} : \rangle \neq 0$. The reason of this *conservation anomaly* stands behind the observation that the parametrix $\boldsymbol{h}$ and $\boldsymbol{w}$ does not separately satisfy the equations of motion [22]. Explicitly, it can be shown that [1]

$$\nabla^a \langle : T_{ab} : \rangle_\omega = \nabla^a \tau^{dD}_{ab} + \nabla^a \tau^{gh}_{ab} \neq 0 \ , \tag{4.11}$$

with τ^{dD}_{ab} and τ^{gh}_{ab} defined as

$$\begin{aligned} \tau^{dD}_{ab} &\doteq \frac{1}{8\pi^2} \left(2 (z^{cd}{}_{cd} - \tfrac{1}{2} z^{c}{}_{c}{}^{d}{}_{d}) g_{ab} - 12 (z^{c}{}_{acb} - \tfrac{1}{2} z^{c}{}_{cab}) \right) \ , \\ \tau^{gh}_{ab} &\doteq \frac{1}{8\pi^2} \left(12 \tilde{z}_{ab} - 4 \tilde{z}_{c}{}^{c} g_{ab} \right) \ . \end{aligned}$$

Here $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$ are the first order contributions to the Taylor series of both $\boldsymbol{v^{(1)}}$ and $\tilde{\boldsymbol{v}}^{\boldsymbol{(1)}}$, as discussed in sections 3.5.3 and 3.5.5. Obviously, a non-conserved tensor is not eligible to be equated with the Einstein tensor $\widetilde{G}_{ab}$, which on its own satisfies the Bianchi identity

$$\nabla^a \widetilde{G}_{ab} = 0 \ .$$

Hence, our current prescription of $\langle : T_{ab} : \rangle$ cannot be used to write the semiclassical Einstein equation (4.2). However, the exact knowledge of the anomalous behaviour of (4.11) allows us to attempt a redefinition of the expectation value of the stress-energy tensor by the subtraction of the non-vanishing contribution of (4.11). Indeed, this modification can be viewed from two different points of view. On one hand, it can be seen as a redefinition of the classical stress-energy tensor by means of a terms which does not alter the canonical on-shell components, while ensuring the covariant conservation after the quantization. On the other hand, the subtraction of the anomaly can be viewed in terms of an operation which preserves the classical stress-energy tensor, while modifying the normal ordering prescription [22].

We now show how to obtain a covariantly conserved quantum stress-energy tensor for linearized gravity. Firstly, we denote the anomalous term as a sum of two contributions

$$\tau_{ab}^{an} \doteq \tau_{ab}^{dD} + \tau_{ab}^{gh} \ ,$$

respectively related to the conservation anomalies of $\langle : T_{ab}^{dD} : \rangle_\omega$ and $\langle : T_{ab}^{gh} : \rangle_\omega$, as we are going to discuss. From the results [1], given by (3.66), (3.67) and (3.73), on a vacuum spacetime background (i.e assuming $R_{ab} = 0$ and $R = 0$) it follows that

$$\tau_{ab}^{dD} = -\frac{1}{8\pi^2}\frac{19}{72} R^{cdef} R_{cdef} g_{ab} \ ,$$
$$\tau_{ab}^{gh} = +\frac{1}{8\pi^2}\frac{11}{720} R^{cdef} R_{cdef} g_{ab} \ .$$

By adding these two contributions, the anomalous term becomes

$$\tau_{ab}^{an} = -\frac{1}{8\pi^2}\frac{179}{720} R^{cdef} R_{cdef} g_{ab} \ . \tag{4.12}$$

Hence, we define the renormalized stress-energy tensor by subtracting the conservation anomaly (4.12) from (4.11)

$$: T_{ab}^{(ren)} : \doteq : T_{ab} : - \ \tau_{ab}^{an} \hat{\mathbb{I}} \ ,$$

thus leading to the expectation value

$$\langle : T_{ab}^{(ren)} : \rangle_\omega = \langle : T_{ab} : \rangle_\omega - \tau_{ab}^{an} \ , \tag{4.13}$$

which is now covariantly conserved

$$\nabla^a \langle : T_{ab}^{(ren)} : \rangle_\omega = 0 \ . \tag{4.14}$$

Since this procedure preserves the regularity of (4.10) while simultaneously ensuring the conservation rule (4.14), we can finally write down the semiclassical Einstein equation as

$$\widetilde{G}_{ab} = 8\pi \langle : T_{ab}^{(ren)} : \rangle_\omega \ , \tag{4.15}$$

with $G = 1$.

In this section we have sketched the Hadamard renormalization procedure, which leads to a finite and covariantly-conserved expectation value of the stress-energy tensor of quantized gravitational radiation. The semiclassical Einstein equation (4.15) allows us to compute the *backreaction* of quantum fields on the background spacetime, provided the choice of a suitable Hadamard quantum state. However, this renormalization scheme, which restores the covariant conservation rule (4.14), still produces a non-negligible outcome. Indeed, to remove τ_{ab}^{an} we have redefined the expectation value (4.10), compensating the conservation anomaly via the subtraction of the non-vanishing term in (4.13). This procedure, while being necessary, definitely modifies the trace of the renormalized expectation value of the stress-energy tensor

$$\langle : T^{(ren)} : \rangle_\omega \doteq g^{ab} \langle : T_{ab}^{(ren)} : \rangle_\omega \ .$$

This leads to the so-called *trace anomaly*. Indeed, by tracing both (4.5) and (4.12) we obtain two different contributions, here resumed as

$$\langle : T^{(ren)} : \rangle_\omega = \langle : T : \rangle_\omega - \tau^{an} \ . \tag{4.16}$$

On one hand, we have a classical trace term, which on a vacuum spacetime (i.e $R_{ab} = 0$ and $R = 0$) reads

$$T_{dD} = \frac{1}{32\pi}(-3\nabla^c \gamma^{ab} \nabla_c \gamma_{ab} + 2\nabla_b \gamma^{ac} \nabla_a \gamma^{cb} - 4\gamma^{ab}\gamma^{cd} R_{acdb}) \ , \tag{4.17}$$

plus the classical contribution coming from ghost fields. On the other hand, we get the trace anomaly term, which in the special case provided by the Schwarzschild (or Kruskal) background gives

$$\tau^{an} = -\frac{179}{30}\frac{1}{\pi^2}\frac{M^2}{r^6} \ . \tag{4.18}$$

Here, we would like to note that the minus sign contained in τ^{an} actually compensates the one of (4.16), leading to a *positive* contribution to the expectation value of the stress-energy tensor.

4.4 Trace anomaly and black hole evaporation

In this section we shall argue that, in the case of a Schwarzschild spacetime describing an (initially static) spherically symmetric black hole, the trace anomaly (4.16) can be used to compute the semiclassical contribution to the Raychaudhuri's equation (4.3), under the choice of a suitable vacuum-like state, such that $[w_{abcd}](x) \simeq 0$ and which allows us to neglect the classical trace contribution given by (4.17).

To clarity extent, we adopt the following shortened notation for the expectation value of the stress-energy tensor

$$\mathcal{T}_{ab} \doteq \langle : T^{(ren)}_{ab} : \rangle_\omega$$

We consider our initial choice of a radial null outgoing geodesic congruence

$$\boldsymbol{k} = f(U,V)\partial_V \ . \tag{4.19}$$

To simplify our future computations, we are going to work with null coordinates (u,v). To apply the result of section (1.4.2), we need to perform a coordinate transformation on (4.19). We recall that, under the map $\{x^\mu \to x^{\mu'}\}$, the components of a vector field $\boldsymbol{k}$ transform as [48, 9]

$$k^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} k^\mu \ .$$

Hence, by applying this to (4.19) and by substitution of (1.16), we get that the only non vanishing component of (4.19) is given by

$$k^v = 4Mfe^{-v/4M} = \frac{4Mf}{V} \ .$$

Under this prescription, the semiclassical contribution to (4.3) actually reads

$$\frac{d\theta}{d\lambda} = -8\pi\varepsilon^2 \mathcal{T}_{vv} k^v k^v + \mathcal{O}(\varepsilon^3) \ . \tag{4.20}$$

By restricting (4.20) to the future horizon $\mathcal{H}^+$, we get

$$\left.\frac{d\theta}{d\lambda}\right|_{\mathcal{H}^+} = -\varepsilon^2 \frac{2\pi\alpha^2 e^2}{M^2}\frac{\mathcal{T}_{vv}}{V^2} + \mathcal{O}(\varepsilon^3) \tag{4.21}$$

with $f|_{\mathcal{H}^+} = -\frac{1}{8}\alpha e M^{-2}$ given by (1.43). It becomes clear that, for this choice of the congruence, the computation of $\mathcal{T}_{vv}$ is actually enough to obtain a measure the derivative of the flux of energy which crosses the event horizon. We shall now discuss the consequences of this observation.

The sign of $\mathcal{T}_{vv}$ helps us to establish whether the black hole evaporates or not. Indeed, for a congruence of radial null outgoing geodesics, the initial conditions

$$\begin{aligned} \theta(\lambda_i)|_{\mathcal{H}^+} &= 0 \ , \\ \frac{d\theta}{d\lambda}(\lambda_i)\bigg|_{\mathcal{H}^+} &= 0 \ , \end{aligned} \tag{4.22}$$

are provided by the observations of section 1.5 on (1.45) and (1.46), assuming the stability of the horizon in the past in absence of quantum gravitational radiation. Hence, the sign of the right-hand side of (4.21) leads to the following statements.

$$\begin{aligned} \left.\frac{d\theta}{d\lambda}\right|_{\mathcal{H}^+} > 0 &\implies \text{increasing outgoing flux of energy from II to I} \ . \\ \left.\frac{d\theta}{d\lambda}\right|_{\mathcal{H}^+} = 0 &\implies \text{no flux of energy} \ . \\ \left.\frac{d\theta}{d\lambda}\right|_{\mathcal{H}^+} < 0 &\implies \text{increasing ingoing flux of energy from II to I} \ . \end{aligned} \tag{4.23}$$

Finally, all that it remains is to complete the computations of the right-hand side of the perturbed Raychaudhuri's equation.

To get (4.21), we present the following discussion from a more general point of view, specializing then to the case of the Kruskal background. Let $(\mathcal{M}, \boldsymbol{g})$ be a spherically symmetric spacetime, whose metric in null coordinates reads

$$ds^2 = -2A(u,v)dudv + R(u,v)^2(d\theta^2 + \sin^2\theta d\varphi^2) \ . \tag{4.24}$$

We choose a suitable quantum state ω, which measures only the stationary spherically symmetric contribution of the backreaction. This implies that

$$\mathcal{T}_{uu}, \ \mathcal{T}_{vv}, \ \mathcal{T}_{uv}, \ \mathcal{T}_{\theta\theta}, \ \mathcal{T}_{\varphi\varphi} \ ,$$

are the only non-vanishing components of the expectation value of the renormalized stress-energy tensor, together with the requirement [11]

$$\mathcal{T}^\theta_{\ \theta} = \mathcal{T}^\varphi_{\ \varphi} \ , \tag{4.25}$$

$$\partial_t \mathcal{T}_{ab} = 0 \ . \tag{4.26}$$

Moreover, we ask that the rotational symmetry of the background spacetime is preserved on the complete one, by fixing the corresponding component of $\mathcal{T}$

$$\mathcal{T}_\theta^\theta = \frac{1}{8\pi} G^\theta{}_\theta \ , \tag{4.27}$$

with G_{ab} the background Einstein tensor. Indeed, through the choice of ω, we are searching for quantum gravitational radiation that is isotropically emitted from the horizon. By applying (4.24) to (4.27), it follows that

$$\mathcal{T}_\theta^\theta = \frac{1}{8\pi}\left(-\frac{1}{A}\partial_u\partial_v \log A - \frac{2}{AR}\partial_u\partial_v R\right) . \tag{4.28}$$

until (4.25) holds, $\mathcal{T}_\theta^\theta$ and $\mathcal{T}_{uv}$ are the only components that actually contributes to the trace (4.16)

$$\mathcal{T} \doteq g^{ab}\mathcal{T}_{ab} = -2\frac{\mathcal{T}_{uv}}{A} + 2\mathcal{T}_\theta^\theta \ . \tag{4.29}$$

The last constraint is provided by the conservation of the stress-energy tensor (4.14), which for (4.24) reads

$$\nabla^a \mathcal{T}_{av} = -\frac{1}{AR^2}\partial_u(\mathcal{T}_{vv}R^2) - \frac{1}{R^2}\partial_v\left(\frac{\mathcal{T}_{uv}}{A}R^2\right) - 2\mathcal{T}_\theta^\theta\,\frac{\partial_v R}{R} = 0 \ . \tag{4.30}$$

Finally, by substitution of (4.29) and (4.28) into (4.30), we obtain a differential equation which allows us to express $\mathcal{T}_{uv}$ in terms of the trace $\mathcal{T}$ and of the components of the background metric (4.24).

We specialize this discussion to the case of a Kruskal spacetime, written in null coordinates (u, v) (1.18), such that [43]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dudv + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \ ,$$

which, for $G = 1$ and $R_s = 2M$, gives

$$A = \frac{1}{2}\left(1 - \frac{2M}{r}\right) \ , \quad R = r$$

By substitution in (4.28), we get $\mathcal{T}_\theta^\theta = 0$, which, together with (4.30) and (4.29), leads to

$$\partial_u(\mathcal{T}_{vv}r^2) = \frac{1}{4}\left(1 - \frac{2M}{r}\right)\partial_v\left(\mathcal{T}r^2\right) \ .$$

By recalling definition (1.14), which gives $u = t - r_*$ and $v = t + r_*$, together with the stationary requirement (4.26), we get

$$\partial_r(\mathcal{T}_{vv}r^2) = -\frac{1}{4}\left(1 - \frac{2M}{r}\right)\partial_r\left(\mathcal{T}r^2\right) \ . \tag{4.31}$$

Here, we have actually exploited property (1.13) of the tortoise coordinate, to easily express the derivative of r_* with respect to r. By recalling the results of the previous section, the trace of the renormalized expectation value of the stress-energy tensor is given by

$$\mathcal{T} = \langle : T : \rangle_\omega - \tau^{an} \ ,$$

with τ^{an} the trace anomaly contribution. We assume that our initial choice of the quantum state ω makes the classical contribution (4.17) negligible, with respect to the anomalous one, giving $\mathcal{T} = -\tau^{an} + \mathcal{O}(\varepsilon)$. Thus, (4.18) reads

$$\mathcal{T} = \frac{179}{30\pi^2}\frac{M^2}{r^6} + \mathcal{O}(\varepsilon) \ .$$

By substitution into (4.31), while taking the derivative with respect to r, we get

$$\partial_r(\mathcal{T}_{vv}r^2) = \frac{179}{30}\frac{M^2}{\pi^2}\left(\frac{1}{r^5} - \frac{2M}{r^6}\right) = 0 \ .$$

Finally, integrating this equation and assuming that, at large distance, the flux of energy related to $\mathcal{T}_{vv}r^2$ vanishes, we obtain that

$$\mathcal{T}_{vv} = \frac{179}{30\pi^2}\frac{M^2}{r^6}\left(-\frac{1}{4} + \frac{1}{5}\frac{2M}{r}\right) \ . \tag{4.32}$$

By substitution into the perturbed Raychaudhuri's equation (4.20), we finally get that

$$\frac{d\theta}{d\lambda} = -\frac{716}{15}\frac{\varepsilon^2}{\pi}\frac{M^2}{r^6}\left(-\frac{1}{4} + \frac{1}{5}\frac{2M}{r}\right)k^v k^v + \mathcal{O}(\varepsilon^3) \ .$$

To conclude our discussion, we evaluate this expression on the future horizon $\mathcal{H}^+$ by taking the limit $r \to 2M$. Given (4.21), we obtain[2]

$$\left.\frac{d\theta}{d\lambda}\right|_{\mathcal{H}^+}^{gr} = \frac{179}{19200}\frac{\alpha^2\varepsilon^2 e^2}{\pi}\frac{1}{M^6}\frac{1}{V^2} + \mathcal{O}(\varepsilon^3) \ , \tag{4.33}$$

which is always positive. By recalling (4.23) we can conclude that the quantum filed $\hat{\gamma}_{ab}$ backreacts on the Kruskal spacetime, inducing the gravitational evaporation of the black hole.

In this section we have argued, see e.g (4.23), that a positive right-hand side of the Raychaudhuri's equation is a sign that the backreaction of the quantum field $\hat{\gamma}_{ab}$ induces the evaporation of the black-hole, by means of an increasing flux of quantum gravitational energy across the event horizon. Indeed, under some physical but reasonable requirement (such as spherical-symmetry, stationarity and the semiclassical Einstein field equations), it has been actually possible to relate the evaporation of a black hole to the presence of a trace anomaly of the renormalized expectation value of the stress-energy tensor of quantized gravitational radiation.

Before ending this section, we would like to discuss result (4.33) as if it was obtained by direct computation of $\langle : \hat{\sigma}^{ab}\hat{\sigma}^{(ren)}_{ab} : \rangle_\omega$, according to our first geometrical approach 1. Indeed, the positivity of (4.33) suggests that the quantum effects produced by $\hat{\gamma}_{ab}$ modify the geometry of the background spacetime, deforming the shape of the congruence of geodesics. As a consequence, they become allowed to cross the event horizon, producing an escape of gravitational energy from the black hole, which is responsible for its actual evaporation. We briefly sketch this situation in figure (4.2).

[2]In view of further results, we are using gr to specify that this contribution to the evaporation is purely gravitational.

From a geometrical point of view, we can recall the relation between θ and the derivative of the cross-sectional area of the congruence A (1.37), which reads

$$\theta = \frac{d}{d\lambda}\log A \ . \tag{4.34}$$

Hence, the positive right-hand side of (4.33) actually has the role of source of monotonic growth for the cross-sectional area of the geodesics, which can break the "trap" provided by the future event horizon $\mathcal{H}^+$ then escaping from the black hole. Indeed, by choosing V as an affine parameter for the congruence, and by taking the derivative of both members of (4.34), we can substitute (4.33), thus getting

$$\frac{d^2}{dV^2}\log A = \frac{d}{dV}\left(\frac{1}{A}\frac{dA}{dV}\right) = \frac{k}{M^6}\frac{1}{V^2} \ , \tag{4.35}$$

with $k \doteq \frac{179}{19200}\frac{\alpha^2 e^2}{\pi}$. To ensure (4.22), we can choose the following initial conditions[3]

$$\begin{aligned} A(V_i) &= A_{\mathcal{H}} \quad \text{for } V \leq V_i \ , \\ \frac{dA}{dV}(V_i) &= 0 \quad \text{for } V < V_i \ , \end{aligned} \tag{4.36}$$

with the area of the event horizon given, at the initial time, by $A_{\mathcal{H}} = 16\pi M^2$. By integrating (4.35) once, we obtain that

$$\frac{1}{A}\frac{dA}{dV} = -\frac{c}{VV_i}(V_i - V) \ ,$$

where we have imposed (4.36) and defined $c \doteq \frac{k}{M^6}$. A further integration, together with the inversion of the log function, leads to the following result

$$A(V) = \begin{cases} 16\pi M^2 \dfrac{V_i^c}{V^c} e^{\frac{c}{V_i}(V-V_i)} & \text{for } V \geq V_i \ , \\ 16\pi M^2 & \text{for } V \leq V_i \ . \end{cases} \tag{4.37}$$

To better understand the behaviour of $A(V)$, we refer to plot 4.1, which has been realized for different values of c and choosing $V_i = 1$. Since c is inversely proportional to the mass of the black hole, we can explicitly argue that: the more massive is the black hole, the slower is the growth of the cross sectional area of the congruence and thus the evaporation of the black hole.

[3]The initial data should be imposed only for $V = V_i$. However, we can actually extend them to the interval $V \leq V_i$, by requiring that the black hole is perturbation-free in the past.

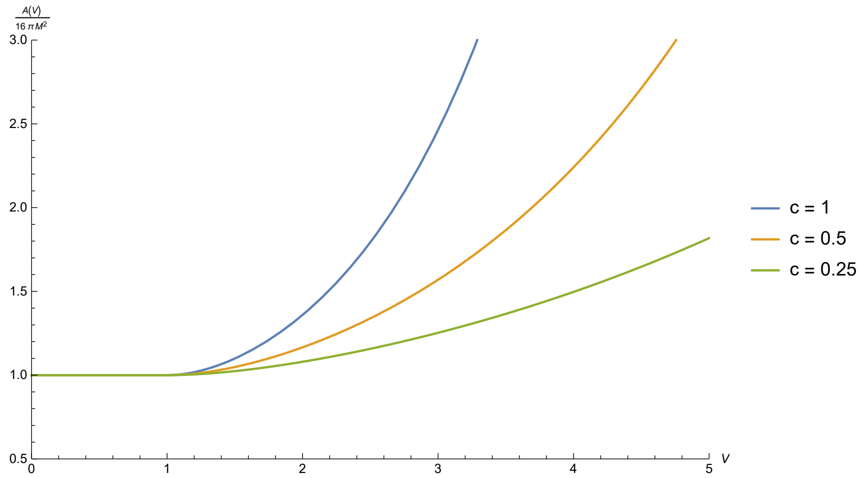
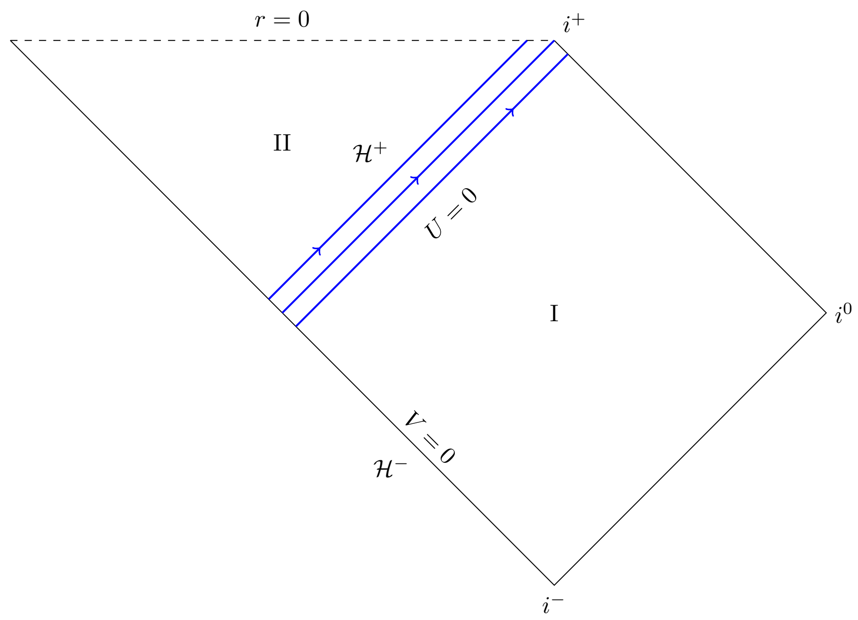


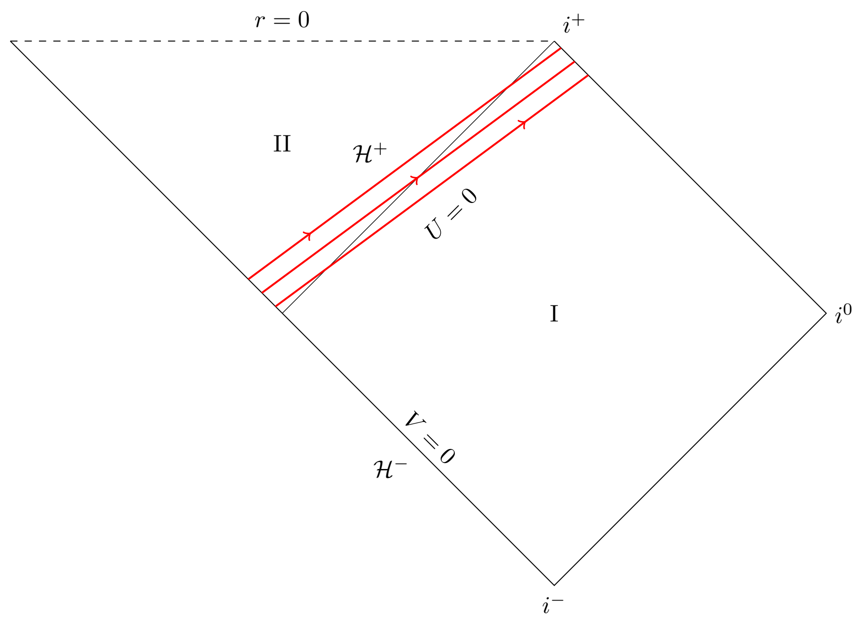
Figure 4.1: Plot of $A(V)$, with initial data at $V_i = 1$ and for $c = 1$, $c = 1/2$ and $c = 1/4$. Realized with Mathematica [29].

However, the answer provided by (4.37) is far from the complete solution, since (4.37) only holds on a right neighbourhood of V_i, where we can neglect the loss of mass of the black hole by the emission of gravitational radiation. In order to achieve a more realistic model, one could try to extend this procedure, accounting for the evaporation with the choice of a different spacetime background. Indeed, by allowing the dependence of the mass from at least one coordinate $M \to M(V)$[4], it would be possible to relate the evolution of the cross-sectional area of the congruence with that of the area of the event horizon $\mathcal{H}$. Therefore, it would be possible to obtain, and hopefully integrate, a local equation for the evolution of the mass M, which unfortunately goes beyond the purpose of this thesis.

[4] Actually, this case is provided by the Vaidya metric [43].



(a) Free outgoing light-like geodesics



(b) Perturbed geodesics

Figure 4.2: Evaporation in terms of a geodesics congruence

4.4.1 From quantum effects on the horizon to Hawking radiation

In this section we consider the previously obtained results, proving that they can be related to the usual interpretation of black hole evaporation in terms of the emission of radiation at a large distance from the horizon, namely *gravitational Hawking radiation.*

We begin our discussion with an observation. Let us consider the Kruskal metric in tortoise coordinates (t, r_*) (1.12), which reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right) dr_*^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \ , \tag{4.38}$$

with r defined implicitly by

$$r_* \doteq r + 2M \log(r/2M - 1) \ ,$$

and giving the event horizon $\mathcal{H}$ by the limit $r_* \to -\infty$ for $r \to 2M$. As in the previous section, we require that the quantum state ω measures only the stationary spherically symmetric contribution of the gravitational backreaction, leading to the following ones as the only non-vanishing components of the expectation value of the stress-energy tensor

$$\mathcal{T}_{tt}, \ \mathcal{T}_{r_*r_*}, \ \mathcal{T}_{tr_*}, \ \mathcal{T}_{\theta\theta}, \ \mathcal{T}_{\varphi\varphi} \ .$$

Moreover, since stationarity and spherical symmetry is preserved, we expect the tensor field $\boldsymbol{\mathcal{T}}$ to depend only from r_* (through r). By the computation of the Christoffel symbols of (4.38), and from the covariant conservation condition (4.14) we get that

$$\nabla^a \mathcal{T}_{at} = \frac{r\left((r - 2M)\partial_{r_*}\mathcal{T}_{tr_*} + 2\mathcal{T}_{tr_*}\right)}{(r - 2M)^2} = 0 \ .$$

By inverting the Leibniz rule, we come to the following condition

$$\partial_{r_*}\left(r^2 \mathcal{T}_{tr_*}\right) = 0 \ ,$$

which states that $r^2\mathcal{T}_{tr_*}$ is constant through the spacetime. Hence, we can measure the flux of quantum gravitational radiation at large distance, i.e the Hawking radiation, by computing $\mathcal{T}_{tr_*}$ directly along the event horizon $\mathcal{H}$

$$r^2 \mathcal{T}_{tr_*}|_{\mathcal{H}} = \lim_{r_* \to \infty} r^2 \mathcal{T}_{tr_*} \ . \tag{4.39}$$

Before entering the computation of the left-hand side of (4.39), we define the *luminosity* of the Hawking radiation of the black hole as [8, 11]

$$\mathcal{L} \doteq -4\pi \lim_{r_* \to \infty} r^2 \mathcal{T}_{tr_*} \ . \tag{4.40}$$

Indeed, this definition guarantees that the presence of outgoing radiation from the horizon, which occurs for $\mathcal{T}_{tr_*}|_{\mathcal{H}} < 0$ (as we shall see in terms of the results of previous section), leads to a positive luminosity.

Let us consider a coordinate transformation $\{x^\mu \to x^{\mu'}\}$, the components of the $(0, 2)$ tensor field $\boldsymbol{\mathcal{T}}$ give [48, 9]

$$\mathcal{T}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \mathcal{T}_{\mu\nu} \ .$$

Hence, by recalling that $u = t - r_*$ and $v = t + r_*$, we get

$$\mathcal{T}_{tr_*} = \mathcal{T}_{vv} - \mathcal{T}_{uu} \ .$$

From definition (1.16), we have that the Kruskal coordinate $U = -e^{-u/2R_s}$, which leads to the following transformation rule

$$\mathcal{T}_{uu} = \frac{U^2}{16M^2}\mathcal{T}_{UU} \ . \tag{4.41}$$

Since the state ω is regular on the horizon $\mathcal{H}$ (located at UV=0), $\mathcal{T}_{UU}|_{\mathcal{H}^+} < +\infty$. The restriction to the hypersurface $\mathcal{H}^+$ can be obtained by evaluating $U = 0$, which, by substitution in (4.41), ensures that

$$\mathcal{T}_{uu}|_{\mathcal{H}^+} = 0 \implies \mathcal{T}_{tr_*}|_{\mathcal{H}^+} = \mathcal{T}_{vv}|_{\mathcal{H}^+} \ .$$

By substitution in definition (4.40), we get the luminosity of gravitational Hawking radiation is given by

$$\mathcal{L}_{gr} = -16\pi M^2 \mathcal{T}_{vv}|_{\mathcal{H}^+} \ . \tag{4.42}$$

In the previous section we have obtained a precise expression for $\mathcal{T}_{vv}$, given in terms of the trace anomaly of quantum gravitational radiation. Exploiting our previous result (4.32) while taking the limit $r \to 2M$, we finally obtain that the luminosity of gravitational Hawking radiation is given by

$$\mathcal{L}_{gr} = \frac{179}{2400}\frac{1}{\pi}\frac{1}{M^2} \ . \tag{4.43}$$

We should note here that this expression of the luminosity is the same obtained by Bekenstein-Hawking (see for instance [39, 40, 41]), except for the numerical factor which depends from the choice of the model describing the evaporation.

Finally, we observe that (4.1) agrees with the prediction of plot (4.1), for which the lighter is the black hole, the greater is its power of emission of Hawking radiation, which translates in a faster loss of mass.

4.4.2 Black hole evaporation from a scalar field

Before ending this chapter we consider the argument of the previous sections, by applying it to the case of the scalar field theory on a curved spacetime background $(\mathcal{M}, \boldsymbol{g})$, whose action is given by

$$S = \int \left(-\tfrac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{2}\xi R\phi^2\right)\sqrt{-g}d^4x \ , \tag{4.44}$$

which leads to the following equations of motion

$$P\phi \doteq \left(\Box - m^2 - \xi R\right)\phi = 0 \ .$$

We will present this discussion in general terms. However, since we are interested in working with a congruence of *null* geodesics, we will later restrict our computation to the massless case. Indeed, all the details regarding the scalar theory, from both the classical and quantum structures to the regularization of Wick monomials has been reviewed in appendix A.

By taking the first variational derivative of (4.44) with respect to the background metric (4.4), we obtain the stress-energy tensor of the scalar theory [22]

$$\begin{aligned} T_{ab}^{KG} = {} & (1-2\xi)\nabla_a\phi\nabla_b\phi - 2\xi(\nabla_a\nabla_b\phi)\phi + \xi G_{ab}\phi^2 + \\ & + g_{ab}\left\{2\xi(\Box\phi)\phi + \left(2\xi - \tfrac{1}{2}\right)\nabla_\rho\phi\nabla^\rho\phi - \tfrac{1}{2}m^2\phi^2\right\} \ , \end{aligned} \tag{4.45}$$

with $\boldsymbol{G}$ the background Einstein tensor. Moreover, the classical trace of (4.45) is given by [22]

$$T \doteq g^{ab}T_{ab}^{KG} = (6\xi - 1)(\phi\Box\phi + \nabla_a\phi\nabla^a\phi) - m^2\phi^2 - \phi P\phi \ ,$$

which vanishes for a massless ($m = 0$), on-shell ($P\phi = 0$), conformally coupled ($\xi = \frac{1}{6}$) scalar field.

From now on, we follow exactly the same discussion of sections 4.2, 4.3 and 4.4. In order to write the semiclassical Einstein equation, and thus computing the backreaction of the field on the background spacetime, we consider an Hadamard quantum state ω, given by (A.22). Hence, the expectation value of the stress-energy tensor is given by $\langle T_{ab}^{KG}\rangle_\omega$. However, expression (4.45) is quadratic in ϕ, hence, its expectation value will be singular in the coinciding point limit. By following the argument adopted for the regularization of the Wick monomials, whose expression for a scalar field is given by (A.29), we regularize $\langle T_{ab}^{KG}\rangle_\omega$ by subtraction of the divergent contribution driven by the Hadamard parametrix (A.23). Indeed, the regularized stress-energy tensor is given by [22]

$$\langle : T_{ab}^{KG} :\rangle_\omega = \frac{1}{8\pi^2}[D_{ab}^{KG}w] \ ,$$

such that

$$\begin{aligned} D_{ab}^{KG}w(x,x') = {} & (1-2\xi)g^{b'}{}_b\nabla_a\nabla_{b'} - 2\xi\nabla_a\nabla_b + \xi G_{ab} + \\ & + g_{ab}\left\{2\xi\Box_x + \left(2\xi - \tfrac{1}{2}\right)g^{c'}{}_c\nabla^c\nabla_{c'} - \tfrac{1}{2}m^2\right\} \ . \end{aligned}$$

As previously argued w and h does not separately satisfy the equation of motion. Hence, our current regularized prescription is not covariantly conserved, being

$$\nabla^a\langle : T_{ab}^{KG} :\rangle_\omega = \nabla^a C_{ab} \neq 0 \ .$$

To be able to write the semiclassical Einstein equation (4.2), we modify our initial definition (4.45), by subtracting the conservation anomaly, encoded by a tensor $\boldsymbol{C}$, such that

$$: T_{ab}^{KG(ren)} : \doteq \, : T_{ab}^{KG} : - \, C_{ab}\hat{\mathbb{I}} \ ,$$

which now gives $\nabla^a\langle : T_{ab}^{KG(ren)} :\rangle_\omega = 0$. As happened for Wick monomials (A.30), the renormalized stress-energy tensor shows multiple degrees of freedom, being always possible to consider the following redefinition [22, 36]

$$: T_{ab}^{KG(ren)}(x)' : = \, : T_{ab}^{KG(ren)}(x) : + \beta_1 m^4 g_{ab} + \beta_2 m^2 G_{ab} + \beta_3 I_{ab} + \beta_4 J_{ab} \ ,$$

with β_1, β_2 and β_3 as renormalization constant and where the tensors $\boldsymbol{I}$ and $\boldsymbol{J}$ are respectively obtained as functional derivative of $\sqrt{g}R^2$ and $\sqrt{g}R^{ab}R_{ab}$.

After this discussion, it is now possible to write down the semiclassical Einstein equation $\tilde{G}_{ab} = 8\pi\langle : T_{ab}^{KG(ren)} :\rangle_\omega$. However, the previous modification to the original prescription of T_{ab} (4.45) introduces an anomalous contribution to the trace of the renormalized expectation value $\langle : T^{KG(ren)} :\rangle_\omega \doteq g^{ab}\langle : T_{ab}^{KG(ren)} :\rangle_\omega$, which now reads [36]

$$\langle : T^{KG(ren)} :\rangle_\omega = \frac{1}{4\pi^2}[v_1] + \frac{1}{8\pi^2}\left(3(\xi - \tfrac{1}{6})\Box - m^2\right)[w] + 4c_1 m^4 - c_2 m^2 R - c_3 \Box R \ ,$$

with c_1, c_2 and c_3 as renormalization constant, and v_1 the Hadamard coefficient given by (A.27). On a vacuum spacetime background (1.6), for a massless and conformally coupled scalar field, this trace anomaly can be computed as

$$\langle : T^{KG(ren)} :\rangle_\omega = \frac{1}{60\pi^2}\frac{M^2}{r^6} \ ,$$

where we have substitute the expression of $[v_1]$ (A.28). Contrary to linearized gravity, the case of a massless conformally coupled scalar field does not provide any classical trace contribution, thus it is not necessary to require $[w] \simeq 0$.

At this point, we can repeat exactly the same procedure of section 4.4. We choose a state ω which preserves the spherical symmetry and stationarity of measurements, then we integrate the covariant conservation relation of the stress-energy tensor, thus relating the trace anomaly with the component

$$\langle : T_{vv}^{KG(ren)} :\rangle_\omega = \frac{1}{60\pi^2}\frac{M^2}{r^6}\left(-\frac{1}{4} + \frac{1}{5}\frac{2M}{r}\right) \ . \tag{4.46}$$

Having imposed the semiclassical Einstein equation (4.2), we substitute this result in the semiclassical Raychaudhuri's equation, finally obtaining that

$$\left.\frac{d\theta}{d\lambda}\right|_{\mathcal{H}^+}^{KG} = \frac{1}{38400}\frac{\alpha^2\varepsilon^2 e^2}{\pi}\frac{1}{M^6}\frac{1}{V^2} + \mathcal{O}(\varepsilon^3) \ .$$

At last, we can derive the contribution of the scalar theory to the luminosity of the black hole. By substitution of (4.46) in definition (4.42), we get that

$$\mathcal{L}_{KG} = \frac{1}{4800\pi}\frac{1}{M^2} \ . \tag{4.47}$$

Conclusions

In this thesis we have considered a model of the evaporation of a (initially static) spherically symmetric black hole. We have started from the theory of linearized gravity to describe the behaviour of quantum gravitational radiation propagating over a Schwarzschild spacetime. Therefore, we have investigated the backreaction of such a quantum field on the background spacetime, which is controlled by the expectation value of the stress-energy tensor, as a consequence of the semiclassical Einstein equation. Indeed, $\langle : T_{ab} : \rangle_\omega$ plays a pivotal role in the Raychaudhuri's equation, where it drives the expansion parameter of a congruence of outgoing null radial geodesics, used to investigate the actual presence of evaporation.

Under some physical assumption on the quantum state ω, we have argued that, on a Schwarzschild spacetime background, the trace anomaly of the stress-energy tensor brings a positive and non-negligible contribution to the right-hand side of the Raychaudhuri's equation, thus leading to a monotonic growth of the flux of outgoing gravitational energy produced by the black hole (4.1). We can interpret this result as follows: the backreaction of the quantum field prevents the event horizon from being able to trap the geodesics within its surface, leading to a violation of the initial stability of the system and thus of a loss of mass of the black hole itself.

Then, we have started from the previous result, considering the conservation of a specific component of $\langle : T_{ab} : \rangle_\omega$. Doing so, we have been able to relate the flux of energy across the horizon with the actual presence of gravitational radiation at large distance from the black hole, namely Hawking radiation, of which we have computed the luminosity (defined as the power of emission of the black hole). In particular, both (4.43) and (4.47) reproduce the expression of the Bekenstein-Hawking luminosity, which depends from the inverse of the square of the mass of the black hole. In view of this, we expect that a suitable forge to test the emission of Hawking radiation is given by primordial black holes, which, if they existed, would be very lightweight, thus leading to a power of emission much more relevant than the one expected from a black hole formed by a stellar collapse.

Finally, we point out that such an evaporation model based on the use of the trace anomaly reveals to be very powerful, since it can be applied to different quantum field theories to describe other types of Hawking radiation on potentially different choices of the spacetime background. Indeed, given ω a suitable quantum state, and T_{ab} the renormalized stress-energy tensor of the chosen theory, its trace-anomaly contribution reads [10, 6]

$$\langle T_\rho{}^\rho \rangle = (2880\pi^2)^{-1} \left(\alpha_1 C_{abcd} C^{abcd} + \alpha_2 (R_{ab} R^{ab} - \tfrac{1}{3} R^2) + \alpha_3 \Box R + \alpha_4 R^2 \right) ,$$

where C_{abcd} is the Weyl tensor, already introduced in (1.34), while α_i are constant which depends from the considered theory and in particular from the coefficients (A, B) of the associated representation of the Lorentz group, resumed in table 4.3.

(A,B)	α_1	α_2	α_3	α_4
$(0,0)$	-1	-1	$(6-30\xi)$	$-90(\xi-\frac{1}{6})^2$
$(\frac{1}{2},0)$	$-\frac{7}{4}$	$-\frac{11}{2}$	3	0
$(\frac{1}{2},\frac{1}{2})$	11	-64	-6	-5
$(1,0)$	-33	27	12	$-\frac{5}{2}$
$(1,\frac{1}{2})$	$\frac{291}{4}$	$\times$	$\times$	$\frac{61}{8}$
$(1,1)$	-189	$\times$	$\times$	$-\frac{747}{4}$

Figure 4.3: Coefficients involved in the computation of the trace anomaly of the stress-energy tensor, with (A, B) labeling the representation of the Lorentz group

From the contents and the results of this work there are different possible insights that can be used to investigate further mathematical and physical details.

Concerning black hole evaporation, it would be interesting to complete the geometrical analysis started in appendix B, performing the renormalization of the squared shear tensor, in order to understand how geodesics paths are modified by the backreaction of the quantum perturbation field. Contrary to the trace anomaly approach, the geometrical point of view does not make use of ghost fields, therefore, one may be interested in a comparison between these results, aimed to understand the role of ghosts in the evaporation process.

In the Klein-Gordon theory it has been already shown that, on a Schwarzschild background, states like the Unruh one [8] can be proved to be of Hadamard [12]. Actually, their choice justifies the computation of the trace anomaly contribution of section 4.4.2. However, this is not the case of the theory of linearized gravity, for which the existence of an Hadamard state satisfying certain properties (such as the stationary requirement) has only been assumed a priori and aimed to mimic the behaviour of the Unruh state of the scalar theory. In order to study the construction of an Hadamard state for linearized gravity, one need first to study its two-point function, whose antisymmetric part is fixed by the causal propagator $\boldsymbol{G}$, associated to the Lichnerowicz equation of motion (2.25). Even if we expect it, a proof of the existence of such a kind of state would confirm our initial ansatz, establishing that it is possible to actually measure the gravitational evaporation (in a quantum mechanical sense).

Finally, it would be very interesting to work with a more general class of spacetimes, removing the static and stationary requirements from the choice of the background. Indeed, by considering a dynamical evolution in the choice of the metric, we could use the Raychaudhuri's equation to relate the growth of the cross-sectional area of the geodesics congruence to the actual area of the horizon of the black hole. In this way, the semiclassical contribution (when driven by the trace anomaly term) would lead to a model predicting the behaviour of the mass of the evaporating black hole, with respect to some time-like coordinate on the spacetime.

Appendices

Appendix A

Review of the free scalar theory

In this chapter we review the classical theory of a scalar field on a curved spacetime, adopting the framework of algebraic quantum field theory.

A.1 Classical theory

We consider the action of a free real scalar field of mass m on a curved spacetime background $(\mathcal{M}, \boldsymbol{g})$, which is given by [22]

$$S = \int \left(-\tfrac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \tfrac{1}{2} m^2 \phi^2 - \tfrac{1}{2} \xi R \phi^2\right) \sqrt{-g} d^4 x \ , \tag{A.1}$$

By taking the variational derivative of (A.1), we get the Einstein-Klein-Gordon equation

$$P\phi \doteq \left(\Box - m^2 - \xi R\right) \phi = 0 \ , \tag{A.2}$$

which describes the dynamics of the field ϕ. Even with a minimally coupled theory, i.e $\xi = 0$, (A.2) is not background independent, since all the information concerning the geometry of the spacetime is carried by the Laplace-Beltrami operator $\Box := g^{\mu\nu} \nabla_\mu \nabla_\nu$.

To study the Cauchy problem associated with (A.2), we briefly review the procedure already discussed in section 2.3.2. However, for a classical scalar field theory, less mathematical details is required, since ϕ is actually a smooth function which takes values on the spacetime

$$\phi : \mathcal{M} \to \mathbb{R} \ .$$

Nevertheless, to connect this prescription to the geometrical language adopted in section A.2, we can define the space of configuration fields ϕ as the space of smooth sections [4]

$$\Gamma(\mathcal{K}) = C^\infty(\mathcal{M}) \ ,$$

where $C^\infty(\mathcal{M})$ is the space of smooth (i.e infinitely differentiable) function on the manifold $\mathcal{M}$, while $\mathcal{K}$ is a vector bundle with base space $\mathcal{M}$ and typical fiber $\mathbb{R}$ (see section 2.3.2 for a definition).

The equation of motion (A.2) can be expressed with respect to the action of a linear partial differential operator P, such that $P\phi = 0$. In the following section we shall discuss the problem of establishing the existence and uniqueness of the solution of P.

Starting from the action, we have characterized the space of configuration of the scalar theory. Now we are ready to discuss the notion of smeared field, which shall play a pivotal role in the algebraic quantization process.

Similarly to what is done in classical mechanics, observables can be defined as functionals on the space of configuration $\Gamma(\mathcal{K})$. From a field theoretical point of view, we may consider some test function $f \in \Gamma_0(\mathcal{K})$, with $\Gamma_0(\mathcal{K}) = C_0^\infty(\mathcal{M})$ the space of compactly supported smooth sections. We define the *smeared classical field* as $F_f(\phi) : \Gamma(\mathcal{K}) \to \mathbb{R}$ such that

$$F_f(\phi) \doteq \int_{\mathcal{M}} \phi(x) f(x) \sqrt{-g} d^4x \ . \tag{A.3}$$

By committing a little abuse of notation, we will denote $F_f(\phi) \doteq \phi(f)$. From a geometrical point of view, prescription (A.3) can be interpreted in terms of a bilinear paring

$$\langle \cdot, \cdot \rangle : \Gamma(\mathcal{K}) \times C_0^\infty(\mathcal{M}) \to \mathbb{R} \ , \quad \phi(f) = \langle \phi, f \rangle \ . \tag{A.4}$$

As long as we consider off-shell configurations, namely those smooth sections which does not satisfy the equation of motion (A.2), the pairing (A.4) is non-degenerate. Indeed, if $\phi(f) = 0$ for all test functions f, then $\phi(x) = 0$. Moreover (A.3) is also injective, since, given two different off-shell field configurations $\phi(x)$, $\psi(x)$, there always exists a test function f such that $\phi(f) \neq \psi(f)$. From these observations we can conclude that $\phi(f)$ is faithfully-labelled by f, hence we can identify $\Gamma_0(\mathcal{K})$ with the *space of off-shell smeared fields*.

Let us now discuss how the dynamical constraint provided by the equation of motion (A.2) on $\Gamma(\mathcal{K})$ is transferred to $\phi(f)$. To this extent, we define

$$\mathrm{Sol}(\mathcal{M}) = \{\phi \in \Gamma(\mathcal{K}) \mid L\phi = 0\} \ ,$$

the *space of on-shell configuration*. Again, we would like to reproduce the previous steps, by defining the space of on-shell classical smeared fields as the set of functionals acting on $\mathrm{Sol}(\mathcal{M})$. It comes trivially from the definition (A.3) that an observable evaluated on some on-shell configuration satisfies the Einstein-Klein-Gordon equation, indeed, if $\phi \in \mathrm{Sol}(\mathcal{M})$ then $(P\phi)(f) = 0$. However, going on-shell introduces also a degeneracy for (A.4) whenever considering test functions of the type $h = Pf$. Integrating by parts, it follows that

$$\phi(Pf) = \langle \phi, Pf \rangle = \langle P\phi, f \rangle = (P\phi)(f) = 0 \ .$$

Hence $\phi(f + Ph) = \phi(f)$ for all $f, h \in \Gamma_0(\mathcal{K})$. Up to this point, test functions does not faithfully label on-shell observables. We overcome this problem by considering the following equivalence classes of test functions

$$[f] = \{f \in \Gamma_0(\mathcal{K}) \mid f = h_1 + Ph_2 \text{ with } h_1, h_2 \in \Gamma_0(\mathcal{K})\} \ .$$

We can recover non-degeneracy of (A.4) and thus a faithfully labelling for $\phi(f)$ by redefining on-shell smeared fields as $\phi([f]) := \langle \phi, f \rangle$, with f any representative of the equivalence class. We conclude this discussion with a definition, which will be useful for further classical discussions. Hence, we call *space of on-shell classical smeared fields*

$$\mathcal{E}(\mathcal{M}) = \Gamma_0(\mathcal{K}) / L\left[\Gamma_0(\mathcal{K})\right] \ .$$

A.1.1 Fundamental solutions of the Einstein-Klein-Gordon equation

Given a spacetime $(\mathcal{M}, g)$, one important question is whether there always exist a solution to equation (A.2) or not. As in the previous chapter, to provide a precise answer to this interrogative, we have to restrict our attention to the class of normally hyperbolic differential operators, thus answering with the following theorem [4, 3, 22].

Theorem 3. *Let $P : \Gamma(\mathcal{K}) \to \Gamma(\mathcal{K})$ be a normally hyperbolic operator on a globally hyperbolic spacetime $(\mathcal{M}, g)$, which is the case provided by the field equation (A.2). Then, the following results hold.*

1. *Let $f \in \Gamma_0(\mathcal{K})$ and Σ be a Cauchy surface of $\mathcal{M}$, with $\boldsymbol{n}$ its future-oriented timelike unit normal vector field. Let $(u_0, u_1) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ be a set of initial data. Then, the Cauchy problem*

$$\begin{cases} P\phi = f \\ \phi|_\Sigma = u_0 \\ \nabla_{\boldsymbol{n}}\phi|_\Sigma = u_1 \end{cases}$$

   has a unique solution $\phi \in \Gamma(\mathcal{K})$. Moreover,

$$\operatorname{supp} u \subset J\left(\mathit{supp} f \cup \operatorname{supp} u_0 \cup \operatorname{supp} u_1\right) \ .$$

   Moreover, the existence and uniqueness is guaranteed also when f, u_0 and u_1 are not compactly supported.

2. *There exist unique retarded E^+ and advanced E^- fundamental solutions of P. Namely, there are unique continuous maps $E^\pm : \Gamma_0(\mathcal{K}) \to \Gamma(\mathcal{K})$, which satisfy*

$$P(E^\pm f) = E^\pm(Pf) = f \ , \quad \operatorname{supp}(E^\pm f) \subset J^\pm_{\mathcal{M}}(\operatorname{supp} f)$$

   for all $f \in \Gamma_0(\mathcal{K})$.

Starting from this result, we can define the *causal propagator* as

$$E \doteq E^- - E^+ \ , \tag{A.5}$$

which satisfies

$$P(Ef) = 0 \ , \quad \operatorname{supp}(E) \subset J_{\mathcal{M}}(\operatorname{supp}(f)) \ .$$

Moreover, the causal propagator satisfies a fundamental property. Given Σ a Cauchy surface, with $\boldsymbol{n}$ its future-oriented timelike unit normal vector field, from the definition it follows that [21]

$$\begin{aligned} &E|_\Sigma = 0 \ , \\ &\nabla_{\boldsymbol{n}} E|_\Sigma = \delta^4 \ . \end{aligned} \tag{A.6}$$

Contrary to the case of linearized gravity, the computation of the propagators of a scalar field is easier to achieve. In the following section we shall see how to determine E from the field equation (A.2), highlighting the differences encountered by adopting two choices of the spacetime background.

A.1.2 Explicit construction of the causal propagator in Minkowski background

Before exploring the curved realm, we shall review the well known case of a scalar field propagating on a static and flat spacetime, namely the Minkowski background. Usually, the computation of the causal propagator is provided through the canonical formalism of creation and annihilation operators. For the following, we shall embrace a distributional point of view, which finds a more suitable generalization to curved spacetimes.

Let us consider a Minkowski background, endowed with the usual metric $\boldsymbol{\eta}$, such that

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

With this particularly choice of the spacetime geometry, the minimally coupled Klein-Gordon equation (A.2) reduces to the flat wave equation

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2\right)\phi = 0 \ ,$$

which leads to the following definition of a linear partial differential operator, driving the dynamics of the classical field

$$P \doteq -\partial_t^2 + \nabla^2 - m^2 \tag{A.7}$$

Being L a normally hyperbolic operator, we compute the retarded and advanced propagator $E^{\pm}$, actually the fundamental bi-solutions of (A.7) whose existence and uniqueness is guaranteed by theorem 3. Writing down the key condition, $PE^{\pm} = \mathrm{id}$, with respect to the distributional kernel of the fundamental solutions, we get that

$$PE^{\pm}(x,y) = \delta^4(x-y) \ . \tag{A.8}$$

This equation can be simplified, exploiting Poincaré invariance and restricting the dependence of $E^{\pm}(x,y)$ to differences of points of the spacetime. Thus, the fundamental solution can be derived with respect to $E^{\pm}(x-y)$, which allows us to Fourier transform (A.8), getting

$$-(k^2+m^2)\hat{E}^{\pm}(k) = 1 \ ,$$

where $k^2 = -k_0^2 + |\vec{k}|^2$. Due to the on-shell singularity, we can't invert the last equation without giving some suitable boundary condition to compute the following integral

$$\int\int_{-\infty}^{+\infty} \frac{e^{ik_0(x^0-y^0)}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{[k_0-\omega(k)]\,[k_0+\omega(k)]} dk_0 d^3k \ , \tag{A.9}$$

where $\omega(k) \doteq |\vec{k}|^2 + m^2$.

Retarded propagator Before computing E^+, we consider its action on test tensors of $\Gamma_0(\mathcal{K})$, which reads in terms of its distributional kernel

$$(E^+f)(x) = \int_{\mathrm{supp}\, f} E^+(x,y)f(y)d^4y \ .$$

We recall that, by theorem 3, $\mathrm{supp}(E^+f) = J^+_{\mathcal{M}}(\mathrm{supp}\, f)$. To make use of this observation, we recall that Poincaré invariance of the Minkowski background guarantees that

$$E^+(x,y) = E^+(x-y)$$

The fundamental property of the $\mathrm{supp}(E^+f)$, brings an additional constraint on the dependence of E^+ from x, y. Indeed, given any point $y \in \mathrm{supp}\, f$, then x need to belong to the future cone of y, which implies that

$$E^+(x-y) = 0 \quad \text{if } x^0 - y^0 < 0 \; . \tag{A.10}$$

Let us consider the following integration, which is involved in (A.9)

$$\mathcal{I} \doteq \int_{-\infty}^{+\infty} \frac{e^{ik_0|x^0-y^0|}}{[k_0 - \omega(k)]\,[k_0 + \omega(k)]} dk_0 \; , \tag{A.11}$$

where we have taken the absolute value of $(x^0 - y^0)$ to ensure property (A.10). In order to choose the integration path, thus taking care of the singularities in $\pm\omega$, we observe that

$$e^{ik_0|x^0-y^0|} \xrightarrow[y\to\pm\infty]{} 0 \quad \text{if } \mathrm{Im}(k_0) > 0 \; .$$

Hence, we can achieve (A.11) by means of the residues theorem [2], which gives a singularities contribution $\mathrm{Re} = i\omega^{-1} \sin\left[\omega(x^0 - y^0)\right]$, leading to

$$\mathcal{I} = -\frac{2\pi}{\omega} \sin\left[\omega(x^0 - y^0)\right] \; .$$

Finally, we can substitute this result in (A.9) thus getting the expression of the retarded propagator

$$E^+(x,y) = \begin{cases} -\dfrac{1}{2\pi^3} \displaystyle\int \frac{1}{\omega} \sin\left[\omega(x^0 - y^0)\right] e^{i\vec{k}\cdot(\vec{x}-\vec{y})} d^3k & \text{if } x^0 - y^0 > 0 \\ \\ 0 & \text{otherwise} \end{cases} \; .$$

Advanced propagator For the advanced fundamental solution E^- we can repeat exactly the same argument of E^+. As far as concerned by $E^-(x-y)$, since $\mathrm{supp}(E^-f) = J^-_{\mathcal{M}}(\mathrm{supp}\, f)$, we have that given any point $y \in \mathrm{supp}\, f$, x need to belong to the future cone of y, thus leading to

$$E^-(x-y) = 0 \quad \text{if } x^0 - y^0 > 0 \; .$$

Mutatis mutandis, the integration (A.9) gives a residue $\mathrm{Re} = -i\omega^{-1} \sin\left[\omega(x^0 - y^0)\right]$, thus obtaining

$$E^-(x,y) = \begin{cases} +\dfrac{1}{2\pi^3} \displaystyle\int \frac{1}{\omega} \sin\left[\omega(x^0 - y^0)\right] e^{i\vec{k}\cdot(\vec{x}-\vec{y})} d^3k & \text{if } x^0 - y^0 < 0 \\ \\ 0 & \text{otherwise} \end{cases} \; .$$

Causal propagator Starting from the previous results, we can exploit definition (A.5), thus obtaining the following expression of the causal propagator

$$E(x,y) = \frac{1}{2\pi^3}\int \frac{1}{\omega}\sin\left[\omega(x^0-y^0)\right] e^{i\vec{k}\cdot(\vec{x}-\vec{y})}d^3k \ .$$

A.1.3 Explicit construction of the causal propagator in Schwarzschild background

We consider a Schwarzschild background spacetime, equipped with the usual metric $\boldsymbol{g}$, such that

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 \ ,$$

where $R_s = 2M$ is the Schwarzschild radius. To further purpose, we recall that this metric is the most general stationary, thus admitting a Killing vector field, associated with time-translation invariance.

With this particularly choice of the metric, the differential operator associated to the minimally coupled Klein-Gordon equation (A.2) becomes

$$P = -\left(1-\frac{R_s}{r}\right)^{-1}\partial_t^2 + \left(1-\frac{R_s}{r}\right)\partial_r^2 - \left(\frac{R_s-2r}{r^2}\right)\partial_r + \frac{L^2}{r^2} - m^2 \ ,$$

where L^2 represents the squared angular momentum operator, defined as

$$L^2 \doteq \frac{1}{\sin^2\theta}\partial_\varphi^2 + \frac{1}{\sin\theta}\left(\sin\theta\partial_\theta\right) \ .$$

We shall compute directly the causal propagator starting from the pivotal property

$$PE = 0 \ , \tag{A.12}$$

which actually states that the distributional kernel $E(x,x')$ is a bi-solution of P, since it satisfies the field equation (A.12) in both arguments. We start our discussion searching for solutions of (A.12) by separation of variables

$$E(x,x') \to A(t,r)\Theta(\theta,\varphi)F(t',\vec{x}') \ .$$

We substitute this prescription in the equation of motion (A.12). By introducing $l(l+1)$ as a separation constant, the angular eigenfunctions equation reads

$$L^2 Y_{lm}(\theta,\varphi) = -l(l+1)Y_{lm}(\theta,\varphi) \ ,$$

where $Y_{lm}(\theta,\varphi)$ are the usual spherical harmonics [2]. On the other hand, the radial equation gives

$$-\left(1-\frac{R_s}{r}\right)^{-1}\partial_t^2 A + \left(1-\frac{R_s}{r}\right)\partial_r^2 A - \left(\frac{R_s-2r}{r^2}\right)\partial_r A = \frac{l(l+1)}{r^2}A + m^2 A \ . \tag{A.13}$$

To simplify our computations, we express (A.13) in terms of the tortoise coordinate r_*, given by (1.12) and which satisfies (1.13). Moreover, by substitution of $u \doteq rA$ in

(A.13), we find that the first radial derivative term vanishes. After this observations, the radial equation finally becomes

$$\left[\partial_t^2 - \partial_{r_*}^2 + V_l(r_*)\right] u_l(t, r_*) = 0 \ , \tag{A.14}$$

where we have introduced a potential V_l as

$$V_l(r_*) \doteq \left(1 - \frac{R_s}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{R_s}{r^3} + m^2\right) \ .$$

We may go a little further, Fourier transforming (A.14) with respect to the Killing time t of $\boldsymbol{g}$, thus obtaining

$$-\partial_{r_*}^2 \hat{u}_l(\omega, r_*) = \left[\omega^2 - V_l(r_*)\right] \hat{u}_l(\omega, r_*) \ .$$

Unfortunately, equation (A.14) does not have any analytical solution, firstly due to the non trivial expression of $r(r_*)$ in terms of the Lambert function [19]. Nevertheless, some useful information about the asymptotic behaviour of u can be deduced, making use of standard scattering theory [20, 21].

Being $E(x, x')$ a bi-solution of (A.12), we can achieve the same results for $F(t', \vec{x}')$. Finally, we can form any bi-solution of (A.12) by taking a linear combination of the angular and radial eigenfunctions. By inverting the Fourier transform, we get

$$E = \int \frac{e^{i(\omega t + \omega t')}}{2\pi\sqrt{\omega\omega'}} \sum_{l,l'} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} C_{ll'}^{mm'} Y_l^m(\theta, \varphi) Y_{l'}^{m'}(\theta', \varphi') \hat{u}_l(\omega, r_*) \hat{u}_{l'}(\omega', r'_*) d\omega d\omega' \ . \tag{A.15}$$

However, this expression is far too difficult to deal with. However, due to the symmetries of the Schwarzschild spacetime together with the properties of the causal propagator, we can simplify it.

Given a bi-distribution $f(t, t')$, we can simplify its Fourier transform $\hat{f}(\omega, \omega')$ by exploiting time-translation invariance. Indeed, from the definition

$$\hat{f}(\omega, \omega') = \frac{1}{2\pi} \int f(t, t') e^{-i\omega t} e^{-i\omega' t'} dt dt' \ . \tag{A.16}$$

If time-translation invariance holds, then $f(t, t') = f(t - t')$. Under the substitution $u = t - t'$, we get

$$\hat{f}(\omega, \omega') = \frac{1}{2\pi} \int f(u) e^{-i\omega u} e^{-i(\omega + \omega')t'} du dt' \ ,$$

which leads to the following relation

$$\hat{f}(\omega, \omega') = \delta(\omega + \omega') \hat{f}(\omega) \ .$$

We consider now the Fourier transform of our bi-solution, such that

$$E(x, x') = \frac{1}{2\pi} \int \hat{E}(\omega, \omega', \vec{x}, \vec{x}') e^{i(\omega t + \omega t')} d\omega d\omega' \ ,$$

with $\hat{E}(\omega, \omega', \vec{x}, \vec{x}')$ given by (A.15). By repeating the same argument already used for (A.16), we obtain that

$$\hat{E}(\omega, \omega', \vec{x}, \vec{x}') = \delta(\omega + \omega')\hat{E}(\omega, \vec{x}, \vec{x}') \ . \tag{A.17}$$

Moreover, since the field ϕ is real, then also $E(x, x')$ does. From a Fourier point of view, this implies that

$$\hat{E}^*(\omega, \vec{x}, \vec{x}') = \hat{E}(-\omega, \vec{x}, \vec{x}') \ . \tag{A.18}$$

By substitution of (A.17) and (A.18), while integrating the Dirac delta with respect to ω', we get

$$E = -i \int \frac{e^{i\omega(t-t')}}{2\pi\omega} \sum_{l,l'} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} C_{ll'}^{mm'} Y_l^m(\theta, \varphi) Y_{l'}^{m'}(\theta', \varphi') \hat{u}_l(\omega, r_*) \hat{u}_{l'}^*(\omega, r_*') d\omega \ . \tag{A.19}$$

Finally, we ask E to be the causal propagator by actually requiring that it satisfies the boundary condition (A.6). To this extent, we replace $\tau \doteq t - t'$ in (A.19). Then, we choose a suitable Cauchy surface Σ, in which the initial data with respect to τ can be considered. Therefore, being E a scalar, equation (A.6) simply reads

$$\partial_\tau E\left(\tau, \vec{x}, \vec{x}'\right)\big|_\Sigma = \frac{1}{r^2}\delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi') \tag{A.20}$$

To make use of this distributional relation, we consider the action of $\partial_\tau E$ on a (real) test function f. Focusing on the angular dependence, we expand $f(\theta, \phi)$ in spherical harmonics

$$f(\theta, \varphi) = \sum f_{pq} Y_p^q(\theta, \varphi) = \sum f_{pq}^* Y_p^{*q}(\theta, \varphi) \ . \tag{A.21}$$

By considering the angular dependence of $\langle \partial_t E, f \rangle$ while integrating the angular Dirac delta in (A.20), we obtain

$$\sum C_{ll'}^{mm'} \int Y_l^m(\theta, \varphi) Y_{l'}^{m'}(\theta', \varphi') f(\theta, \varphi) d\Omega = f(\theta', \varphi') \ .$$

By substitution of (A.21) in both members, while exploiting the orthonormality of spherical harmonics $\int Y_l^m Y_p^{*q} d\Omega = \delta_q^m \delta_p^l$, we get

$$\sum C_{ll'}^{mm'} Y_{l'}^{m'}(\theta', \varphi') f_{lm}^* = f(\theta', \varphi') \ .$$

Expanding one more time the right-hand side and equating both members "mode by mode", we get the fundamental property

$$\sum C_{ll'}^{mm'} Y_{l'}^{m'}(\theta', \varphi') = Y_l^{*m}(\theta', \varphi') \ .$$

Actually this result can also be extended by considering test functions which also depend from r. By exploiting the completeness relation of $\hat{u}(\omega, r_*)$ [21, Ch. 2]

$$\sum_{l=0}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}_l(\omega, r_*) \hat{u}_l^*(\omega, r_*') d\omega = \frac{1}{r^2}\delta(r - r') \ ,$$

and thus repeating the former argument for $f(r, \theta, \varphi)$, we get that

$$\sum C_{ll'}^{mm'} Y_{l'}^{m'}(\theta', \varphi')\hat{u}_l(\omega, r_*)\hat{u}_{l'}^*(\omega, r'_*) = Y_l^{*m}(\theta', \varphi')\hat{u}_l(\omega, r_*)\hat{u}_l^*(\omega, r'_*) \ .$$

Finally, we can substitute this result in (A.19), getting the following expression of the causal propagator [8]

$$E = -i \int \frac{e^{i\omega(t-t')}}{2\pi\omega} \sum_{l,m} Y_l^m(\theta, \varphi) Y_l^{*m}(\theta, \varphi)\hat{u}_l(\omega, r_*)\hat{u}_{l'}^*(\omega, r'_*)d\omega \ .$$

A.2 Quantum theory

Once that the classical theory is known, the quantization of the Klein-Gordon field on a curved spacetime background $(\mathcal{M}, \boldsymbol{g})$ can be achieved by means of the algebraic framework, where the causal propagator E plays a pivotal role in encoding the dynamical information of the field equation (A.2) in the canonical commutation rules.

Contrary to the case of linearized gravity, the action (A.1) has no symmetry which requires any treatment via BRS formalism. Hence, the quantization can be achieved by promoting the smeared field to an element of a unital $*$-algebra $\mathcal{A}(\mathcal{M})$, with unity $\hat{\mathbb{I}}$, which satisfies [31, 4]

1. **Linearity**: $\hat{\phi}(c_1 f_1 + c_2 f_2) = c_1\hat{\phi}(f_1) + c_2\hat{\phi}(f_2)$, with $c_1, c_2 \in \mathbb{C}$.
2. **Hermiticity**: $\hat{\phi}(f)^* = \hat{\phi}(f^*)$.
3. **Symmetry**: $\hat{\phi}(f) = 0$, for any anti-symmetric $f \in \Gamma_0(\mathcal{V})$.
4. **Field equations**: $\hat{\phi}(Pf) = 0$.
5. **Commutation relation**: $[\hat{\phi}(f), \hat{\phi}(f')] = iE(f, f')\hat{\mathbb{I}}$.

Here f is any test tensor of $\Gamma_0(\mathcal{K})$. Hence we call *observable* any element $O \in \mathcal{A}(\mathcal{M})$, written as a polynomial of the generators of the algebra [31], such that

$$\hat{O}^* = \hat{O} \ .$$

Once that abstract structure of the algebra of fields is know, we can define a *state* ω as any positive and normalized functional [31]

$$\omega : \mathcal{A}(\mathcal{M}) \to \mathbb{C} \ ,$$

which allows to compute the expectation value of any observable of the algebra $\langle O \rangle_\omega$.

Again, we are interested in working with those state whose two-point correlation function $\omega(x, x') \doteq \langle\hat{\phi}(x)\hat{\phi}(x')\rangle_\omega$, which defined as

$$\omega(\hat{\phi}(f)\hat{\phi}(f')) \doteq \int \omega(x, x')f(x)f'(x')\sqrt{-g}\sqrt{-g'}d^4x d^4x' \ ,$$

has a universal divergent behaviour. Indeed, we call *Hadamard quasi-free state*, any state ω, such that $\omega(x, x')$ satisfies

$$\omega(x, x') = \lim_{\epsilon \downarrow 0} \frac{1}{8\pi^2}\left(h^\epsilon(x, x') + w(x, x')\right) \ , \tag{A.22}$$

with w a symmetric bi-scalar regular on the coinciding point limit, which encodes the freedom in the choice of the state ω, and h the *Hadamard parametrix* given by

$$h^{\epsilon}(x,x') \doteq \frac{u(x,x')}{\sigma_{\epsilon}(x,x')} + v(x,x')\log\left(\frac{\sigma_{\epsilon}(x,x')}{\lambda^2}\right) \ , \tag{A.23}$$

which, on the other hand, describes the divergent behaviour of ω. Here σ is half the squared geodesic distance (3.48) between x and x', while u and v are two smooth bi-scalar.

The structure of the classical theory, encoded by P and stored in ω, arises when recalling that the two-point correlation function of a quasi-free state, which completely describes ω by the Wick theorem [31], satisfies

$$\begin{aligned} &\omega(f,f') - \omega(f',f) = iE(f,f') \ , \\ &\omega(Pf,f') = \omega(f,Pf') = 0 \ . \end{aligned} \tag{A.24}$$

Moreover, we can repeat the same argument of section (3.5.2), expanding v in series of σ

$$v(x,x') = \sum_{n=0}^{\infty} v^{(n)}(x,x')\sigma^n \ .$$

Indeed, we impose the equation of motions $P(h+w)=0$ (A.24) while requiring that the singular terms of h cancel each other out, order by order in σ. Again, the result of this discussion leads to the Hadamard recursive relations [22]

$$2\nabla_\mu\sigma\nabla^\mu u + (\Box - 4)u = 0 \ ,$$

$$Pu + \Box\sigma v^{(0)} - 2v^{(0)} + 2\nabla_\mu\sigma\nabla^\mu v^{(0)} = 0 \ , \tag{A.25}$$

$$P\tilde{v}^{(n)} + 2(n+1)\nabla_\mu\sigma\nabla^\mu\tilde{v}^{(n+1)} + (n+1)(\Box\sigma + 2n)\tilde{v}^{(n+1)} = 0 \ . \tag{A.26}$$

These equations can be solved, upon the choice of suitable initial conditions. Usually, in order to recover the limit of the Minkowski spacetime, one requires that $[u]=1$[1], allowing u to be expressed in term of the Van Vleck-Morette determinant (3.63) [22, 34]. By exploiting this data while taking the coinciding point limit of (A.25) and (A.26) we obtain that

$$\begin{aligned} [v^{(0)}] &= \frac{1}{2}[Pu] \ , \\ [v^{(n+1)}] &= \frac{1}{2(n+1)(n+2)}[Pv^{(n)}] \ . \end{aligned}$$

Indeed, the v_0 coefficient satisfies [23]

$$[v_0] = \frac{1}{2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right] \ ,$$

while the v_1 coefficient reads

$$\begin{aligned} [v_1] = \frac{m^4}{8} + \frac{(6\xi-1)m^2R}{24} + \frac{(6\xi-1)m^2R}{288} + \\ + \frac{(1-5\xi)\Box R}{120} - \frac{R_{ab}R^{ab}}{720} + \frac{R_{abcd}R^{abcd}}{720} \ . \end{aligned} \tag{A.27}$$

[1] As in the rest of this thesis, we are denoting the coinciding point limit as $[u] \doteq \lim_{x'\to x} u(x,x')$.

Since these two objects are scalars, we compute them for a specific choice of the metric $\boldsymbol{g}$, obtaining then a results which holds on any spacetime. By choosing the Schwarzschild background (4.38), for a massless and conformally coupled scalar field, it follows that

$$\begin{aligned} [v_0] &= 0 \ , \\ [v_1] &= \frac{48}{720}\frac{M^2}{r^6} \ , \end{aligned} \tag{A.28}$$

with M the mass of the black hole, and r the radial coordinate.

Once that the behaviour of Hadamard states is known, we can extend the algebra $\mathcal{A}(\mathcal{M})$ to include also monomials like $\hat{\phi}^2$, by regularizing the two-point correlation function, through the subtraction of the divergences encoded by the Hadamard parametrix (A.23) via the so-called point-splitting prescription [22]

$$:\hat{\phi}(x)\hat{\phi}(x'): \doteq \hat{\phi}(x)\hat{\phi}(x') - \frac{1}{8\pi^2}h(x,x')\hat{\mathbb{I}} - H(x,x')\hat{\mathbb{I}} \ ,$$

with $H(x,x')$ any bi-scalar, which has finite coinciding point limit. According to this procedure, the expectation value of $\hat{\phi}^2$ gives

$$\langle :\hat{\phi}(x)\hat{\phi}(x):\rangle_\omega = \frac{1}{8\pi^2}[w] - [H] \ . \tag{A.29}$$

More details about this machinery can be found in section 3.5.4. Before ending this discussion, we point out that the degrees of freedom brought by $H(x,x')$ cannot be neglected. Indeed, under the requirement of covariance and locality, they can be classified in the following way [36]

$$:\hat{\phi}^2(x)': \,=\, :\hat{\phi}^2(x): + \left(\alpha_1 R + \alpha_2 m^2\right)\hat{\mathbb{I}} \ , \tag{A.30}$$

with α_1, α_2 two arbitrary renormalization constants.

Appendix B

The geometrical approach

In this section we discuss the geometrical contribution to the perturbed Raychaudhuri's equation (4.1). We start again from the linearization of the complete metric (2.1), being

$$\tilde{g}_{ab} = g_{ab} + \varepsilon\gamma_{ab} + \mathcal{O}(\varepsilon^2) \ , \tag{B.1}$$
$$\tilde{g}^{ab} = g^{ab} - \varepsilon\gamma^{ab} + \mathcal{O}(\varepsilon^2) \ . \tag{B.2}$$

Again we are adopting the convention of [48], actually using the background metric g_{ab} to raise and lower the indices. We consider the effect related to the presence of gravitational radiation by means of an expansion of the fundamental quantities related to the congruence, namely the tangent vector field $\boldsymbol{k}$ and the deviation tensor $\boldsymbol{B}$

$$k^a \to k^a + \varepsilon\xi^a \ ,$$
$$B_{ab} \to B_{ab} + \varepsilon\mathcal{B}_{ab} \ ,$$

as a consequence of prescription (B.1) and (B.2). During sections B.1 and B.2 we shall investigate the relation between $\boldsymbol{\xi}$, $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\gamma}$.

B.1 Perturbations and geodesics

In order to understand the geometrical contribution to (4.1), it is necessary to study the role of the modification to the geodesics paths, induced by the gravitational perturbation $\boldsymbol{\gamma}$. This can be done by means of a perturbative analysis of the geodesic equation on the complete spacetime $(\widetilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$.

We consider two vector fields $\tilde{\boldsymbol{k}}, \boldsymbol{k}$, which respectively describe the congruences associated to the complete and background spacetimes.

As stressed in the first chapter, we model the static spherically symmetric black hole by endowing the background spacetime with the Kruskal metric (1.18). Moreover, we restrict our attention to a congruence of radial null geodesics $\boldsymbol{k}$, which in null coordinates is given by (1.39).

We consider a linear expansion of the congruence tangent vector field, which, under prescription (B.1), gives

$$\tilde{k}^a \doteq k^a + \varepsilon\xi^a + \mathcal{O}(\varepsilon^2) \ . \tag{B.3}$$

According to our initial choice (1.39), $\boldsymbol{k}$ is a null geodesic on the background spacetime. We assume that any gravitational perturbation preserves this fundamental property, by actually requiring that

$$\tilde{g}_{ab}\tilde{k}^a\tilde{k}^b = 0 \ .$$

By substitution of (B.3), we isolate the zeroth and first order terms, respectively getting

$$g_{ab}k^a k^b = 0 \ , \tag{B.4}$$

$$g_{ab}\xi^a k^b + \gamma_{ab}k^a k^b = 0 \ . \tag{B.5}$$

Actually, property (B.4) is nothing but the null requirement for the contribution of the congruence on the background spacetime $(\mathcal{M}, \boldsymbol{g})$, which is preserved as perturbation-free limit. Moreover, we ask $\widetilde{\boldsymbol{k}}$ to satisfy the geodesics equation on the complete spacetime $(\widetilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, which naturally reads as

$$\widetilde{k}^a\widetilde{\nabla}_a\widetilde{k}^b = 0 \ .$$

We expand this equation, with respect to (B.3) and the expression of the complete covariant derivative (2.2). The zeroth order term reduces to the background geodesic equation $k^a\nabla_a k^b = 0$, while at first order we get

$$k^a\nabla_a\xi^c + \xi^a\nabla_a k^c + C^c{}_{ab}k^a k^b = 0 \ ,$$

By substitution of (2.8), we get

$$k^a\nabla_a\xi^c + \xi^a\nabla_a k^c + k_a k_b\nabla^a\gamma^{bc} - \tfrac{1}{2}k_a k_b\nabla^c\gamma^{ab} = 0 \ . \tag{B.6}$$

Once that the initial condition for $\boldsymbol{\xi}$ is given, that is $\xi = 0$ in the past when there is no gravitational radiation, we can solve (B.6) to uniquely express the first order contribution $\boldsymbol{\xi}$ in terms of the perturbation field $\boldsymbol{\gamma}$.

An additional constraint on $\boldsymbol{\gamma}$ can be obtained by contracting (B.6) with k^b. Indeed, properties (B.4) and (B.5), together with the background geodesic equation, give

$$k^c\nabla_c(k_a k_b\gamma^{ab}) = k_a k_b k^c\nabla_c\gamma^{ab} = 0 \ .$$

Hence, the scalar $k_a k_b\gamma^{ab}$ is constant along the background geodesics congruence.

B.2 Perturbations and the squared shear tensor

In order to get an expression for $\hat{\sigma}^{(1)}_{ab}\hat{\sigma}^{ab}_{(1)}$ to in (4.1), we consider the effect of $\boldsymbol{\gamma}$ on the deviation tensor $\boldsymbol{B}$, given by (1.20). As for the congruence vector field $\boldsymbol{k}$, we model the effect of the perturbation $\boldsymbol{\gamma}$ by means of the following linear expansion

$$\widetilde{B}_{ab} = B_{ab} + \varepsilon\mathcal{B}_{ab} + \mathcal{O}(\varepsilon^2) \ .$$

Indeed, we start from the definition (1.20), which on the complete spacetime reads

$$\widetilde{B}_{ab} = \tilde{g}_{ac}\tilde{\nabla}_b\tilde{k}^c \ .$$

Expanding this definition with respect to (B.3), (2.1) and (2.2), we get

$$\widetilde{B}_{ab} = \nabla_b k_a + \varepsilon(\nabla_b \xi_a + \gamma_{ac}\nabla_b k^c + g_{ac}C^c{}_{bd}k^d) + \mathcal{O}(\varepsilon^2) \ . \tag{B.7}$$

Before exploiting this prescription to quantify the effect of gravitational radiation, we firstly consider the transverse component of $\tilde{\boldsymbol{B}}$, by means of the projector $\tilde{\boldsymbol{h}}$ (1.27), which on the complete spacetime reads

$$\tilde{h}_a{}^b = \delta_a{}^b + \tilde{g}_{ac}\tilde{k}^c\tilde{l}^b + \tilde{g}_{ac}\tilde{l}^c\tilde{k}^b \ , \tag{B.8}$$

with $\tilde{g}_{ab}\tilde{k}^a\tilde{l}^b = -1$. By simply translating (1.28) to the complete spacetime notation, we get

$$\tilde{k}^a\tilde{h}_a{}^b = 0 \ , \quad \tilde{g}_{bc}\tilde{k}^b\tilde{h}_a{}^c = 0 \ . \tag{B.9}$$

Again, we consider the definition of $\hat{\boldsymbol{B}}$ (1.26), which on $(\widetilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ reads

$$\hat{\widetilde{B}}_{ab} = \tilde{h}_a{}^c\tilde{h}_b{}^d\widetilde{B}_{cd} \ . \tag{B.10}$$

Since $\tilde{\boldsymbol{k}}$ satisfies both the null condition and the geodesic equation, property (1.21) can be translated to the complete spacetime as

$$\tilde{k}^a\widetilde{B}_{ab} = \tilde{k}^a\widetilde{B}_{ba} = 0 \ .$$

We exploit this property to simplify the computation of $\hat{\boldsymbol{\sigma}}$. Indeed, a partial substitution of (B.8) in (B.10) gives

$$\tilde{h}_b{}^d\widetilde{B}_{cd} = (\delta_b{}^d + \tilde{g}_{be}\tilde{k}^e\tilde{l}^d + \tilde{g}_{be}\tilde{l}^e\tilde{k}^d)\widetilde{B}_{cd} = (\delta_b{}^d + \tilde{g}_{be}\tilde{l}^e\tilde{k}^d)\widetilde{B}_{cd} \ ,$$

by multiplying once more, we get

$$\hat{\widetilde{B}}_{ab} = (\delta_a{}^c + \tilde{g}_{ae}\tilde{k}^e\tilde{l}^c + \tilde{g}_{ae}\tilde{l}^e\tilde{k}^c)\tilde{h}_b{}^d\widetilde{B}_{cd} = (\delta_a{}^c + \tilde{g}_{ae}\tilde{l}^c\tilde{k}^e)(\delta_b{}^d + \tilde{g}_{bf}\tilde{l}^d\tilde{k}^f)\widetilde{B}_{cd} \ .$$

Before giving further details, we recall the properties of the shear tensor. In section 1.4 we have discussed what happens when considering a Kruskal background, which from (1.30) gives

$$\hat{\sigma}_{ab}|_{\mathcal{H}^+} = \hat{\widetilde{B}}_{(ab)} + \mathcal{O}(\varepsilon^2) \ , \quad \hat{\sigma}^{(0)}_{ab}|_{\mathcal{H}^+} = 0 \ ,$$

and then

$$\hat{\sigma}_{ab}\hat{\sigma}^{ab}|_{\mathcal{H}^+} = \frac{d}{d\varepsilon^2}(\hat{\widetilde{B}}_{(ab)}\hat{\widetilde{B}}^{ab})\Big|_{\varepsilon=0} + \mathcal{O}(\varepsilon^3) \ . \tag{B.11}$$

To this extent we compute the squared deviation tensor, which, by a partial substitution of the previous results, reads

$$\begin{aligned}\hat{\widetilde{B}}_{(ab)}\hat{\widetilde{B}}^{ab} &= \tilde{g}^{al}\tilde{g}^{bm}\tilde{h}_l{}^e\tilde{h}_m{}^f\widetilde{B}_{cd}\widetilde{B}_{ef}(\delta_{(a}{}^c + \tilde{g}_{e(a}\tilde{l}^c\tilde{k}^e)(\delta_{b)}{}^d + \tilde{g}_{b)f}\tilde{l}^d\tilde{k}^f) = \\ &= \tilde{g}^{al}\tilde{g}^{bm}\tilde{h}_l{}^e\tilde{h}_m{}^f\widetilde{B}_{cd}\widetilde{B}_{ef}(\delta_{(a}{}^c\delta_{b)}{}^d + \tilde{g}_{e(a}\delta_{b)}{}^d\tilde{l}^c\tilde{k}^e + \delta_{(a}{}^c\tilde{g}_{b)f}\tilde{l}^d\tilde{k}^f + \tilde{g}_{e(a}\tilde{g}_{b)f}\tilde{l}^d\tilde{k}^f\tilde{l}^c\tilde{k}^e)\end{aligned}$$

We employ property (B.9), which makes the last three terms vanish (being contracted with $\tilde{\boldsymbol{h}}$). Exploiting the Kronecker delta, we get that

$$\hat{\widetilde{B}}_{(ab)}\hat{\widetilde{B}}^{ab} = \tilde{g}^{al}\tilde{g}^{bm}\tilde{h}_l{}^e\tilde{h}_m{}^f\widetilde{B}_{(ab)}\widetilde{B}_{ef} \ .$$

Again, property (B.9) hits both the deviation tensor contributions, neglecting the $\tilde{\boldsymbol{k}}$ terms contained in $\tilde{\boldsymbol{h}}$, finally giving

$$\hat{\tilde{B}}_{(ab)}\hat{\tilde{B}}^{ab} = \tilde{g}^{ac}\tilde{g}^{bd}\tilde{B}_{(ab)}\tilde{B}_{cd} \ .$$

From this last result we can obtain an expression for $\hat{\sigma}^{(1)}_{ab}\hat{\sigma}^{ab}_{(1)}$, by actually substituting (B.7) and then isolating the second order terms, according to observation (B.11). Indeed, we obtain that

$$\hat{\tilde{B}}_{(ab)}\hat{\tilde{B}}^{ab} = (g^{ac}-\varepsilon\gamma^{ac})(g^{bd}-\varepsilon\gamma^{bd})[\nabla_{(b}k_{a)}+\varepsilon(\nabla_{(b}\xi_{a)}+\gamma_{e(a}\nabla_{b)}k^e+g_{e(a}C^e{}_{b)f}k^f)]\tilde{B}_{cd} \ .$$

In particular, we consider the first three factors, which give

$$\begin{aligned}
&(g^{ac}-\varepsilon\gamma^{ac})(g^{bd}-\varepsilon\gamma^{bd})[\nabla_{(b}k_{a)}+\varepsilon(\nabla_{(b}\xi_{a)}+\gamma_{e(a}\nabla_{b)}k^e+g_{e(a}C^e{}_{b)f}k^f)] = \\
&= \nabla^{(d}k^{c)}+\varepsilon[\nabla^{(d}\xi^{c)}+\gamma_e{}^{(c}\nabla^{d)}k^e+g^{ac}g^{bd}g_{e(a}C^e{}_{b)f}k^f-\gamma_a{}^c\nabla^{(d}k^{a)}-\gamma_b{}^d\nabla^{(b}k^{c)}]+ \\
&\quad +\varepsilon^2[\gamma^{ac}\gamma^{bd}\nabla_{(b}k_{a)}-\gamma_a{}^c\nabla^{(d}\xi^{a)}-\gamma_a{}^c\gamma_e{}^{(a}\nabla^{d)}k^e-\gamma^{ac}g^{bd}g_{e(a}C^e{}_{b)f}k^f+ \\
&\qquad -\gamma_b{}^d\nabla^{(b}\xi^{c)}-\gamma_b{}^d\gamma_e{}^{(c}\nabla^{b)}k^e-\gamma^{bd}g^{ac}g_{e(a}C^e{}_{b)f}k^f]+\mathcal{O}(\varepsilon^3) \ .
\end{aligned}$$

By substitution into the expansion of $\hat{\tilde{B}}_{(ab)}\hat{\tilde{B}}^{ab}$, we get that

$$\begin{aligned}
\hat{\tilde{B}}_{(ab)}\hat{\tilde{B}}^{ab} &= \Big\{\nabla^{(d}k^{c)}+\varepsilon\big[\nabla^{(d}\xi^{c)}+\gamma_e{}^{(c}\nabla^{d)}k^e+g^{ac}g^{bd}g_{e(a}C^e{}_{b)f}k^f-\gamma_a{}^c\nabla^{(d}k^{a)}+ \\
&-\gamma_b{}^d\nabla^{(b}k^{c)}\big]+\varepsilon^2\big[\gamma^{ac}\gamma^{bd}\nabla_{(b}k_{a)}-\gamma_a{}^c\nabla^{(d}\xi^{a)}-\gamma_a{}^c\gamma_e{}^{(a}\nabla^{d)}k^e-\gamma^{ac}g^{bd}g_{e(a}C^e{}_{b)f}k^f \\
&-\gamma_b{}^d\nabla^{(b}\xi^{c)}-\gamma_b{}^d\gamma_e{}^{(c}\nabla^{b)}k^e-\gamma^{bd}g^{ac}g_{e(a}C^e{}_{b)f}k^f\ \big]\Big\}\Big\{\nabla_dk_c+\varepsilon\big[\nabla_d\xi_c+\gamma_{gc}\nabla_dk^g+ \\
&\qquad +g_{hc}C^h{}_{di}k^i\ \big]\Big\}
\end{aligned}$$

According to (B.11), we isolate the second order terms, thus getting

$$\begin{aligned}
\hat{\sigma}_{ab}\hat{\sigma}^{ab}|_{\mathcal{H}^+} &= \nabla^{(d}\xi^{c)}\nabla_d\xi_c+\gamma_e{}^{(c}\nabla^{d)}k^e\nabla_d\xi_c+g_{e(a}C^e{}_{b)f}k^f\nabla^b\xi^a-\gamma_a{}^c\nabla^{(d}k^{a)}\nabla_d\xi_c+ \\
&-\gamma_b{}^d\nabla^{(b}k^{c)}\nabla_d\xi_c+\nabla^{(d}\xi^{c)}\gamma_{gc}\nabla_dk^g+\gamma_e{}^{(c}\nabla^{d)}k^e\gamma_{gc}\nabla_dk^g+g_{e(a}C^e{}_{b)f}k^f\gamma_g{}^a\nabla^bk^g+ \\
&-\gamma_a{}^c\nabla^{(d}k^{a)}\gamma_{gc}\nabla_dk^g-\gamma_b{}^d\nabla^{(b}k^{c)}\gamma_{gc}\nabla_dk^g+\nabla^{(d}\xi_{h)}C^h{}_{di}k^i+\gamma_{e(h}\nabla^{d)}k^eC^h{}_{di}k^i+ \\
&+g^{bd}g_{e(a}C^e{}_{b)f}k^fC^a{}_{di}k^i-\gamma_{ah}\nabla^{(d}k^{a)}C^h{}_{di}k^i-\gamma^{bd}\nabla_{(b}k_{h)}C^h{}_{di}k^i+\gamma^{ac}\gamma^{bd}\nabla_{(b}k_{a)}\nabla_dk_c+ \\
&-\gamma_a{}^c\nabla^{(d}\xi^{a)}\nabla_dk_c-\gamma_a{}^c\gamma_e{}^{(a}\nabla^{d)}k^e\nabla_dk_c-\gamma^{ac}g_{e(a}C^e{}_{b)f}\nabla^bk_ck^f-\gamma_b{}^d\nabla^{(b}\xi^{c)}\nabla_dk_c+ \\
&-\gamma_b{}^d\gamma_e{}^{(c}\nabla^{b)}k^e\nabla_dk_c-\gamma^{bd}g_{e(a}C^e{}_{b)f}k^f\nabla_dk^a \ ,
\end{aligned}$$

with the crossed symmetrization defined as

$$\nabla^{(d}\xi_{h)} \doteq \tfrac{1}{2}(\nabla^d\xi_h+\nabla_h\xi^d)$$

We isolate the different terms of the second last expression, searching for some simplification. By resolving the symmetrization of all the different contributions we get

$$\begin{aligned}
\hat{\sigma}_{ab}\hat{\sigma}^{ab}|_{\mathcal{H}^+} &= \tfrac{1}{2}\nabla^{(d}\xi^{c)}\nabla_d\xi_c+\nabla^d\xi_hC^h{}_{di}k^i+\nabla_h\xi^dC^h{}_{di}k^i-\gamma^{bd}\nabla_dk_eC^e{}_{bf}k^f+ \\
&-\tfrac{1}{2}\gamma^{ac}\nabla_ek_cC^e{}_{af}k^f-\tfrac{1}{2}\gamma_{ed}\nabla^dk^aC^e{}_{af}k^f-\gamma_b{}^d\nabla^b\xi^c\nabla_dk_c-\gamma_b{}^d\nabla^c\xi^b\nabla_dk_c+ \\
&-\gamma_a{}^c\gamma_c{}^g\nabla^ak^d\nabla_dk_g-\tfrac{1}{2}\gamma^c{}_a\gamma^a{}_e\nabla^dk^e\nabla_dk_c-\tfrac{1}{2}\gamma_{bd}\gamma_{ec}\nabla^bk^e\nabla^dk^c+g^{bd}g_{e(a}C^e{}_{b)f}k^fC^a{}_{di}k^i
\end{aligned}$$

By substitution of (2.8), we obtain that

$$\begin{aligned} \nabla^d \xi_h C^h{}_{di} k^i + \nabla_h \xi^d C^h{}_{di} k^i &= \nabla^m \xi^d k^i \nabla_i \gamma_{md} \ , \\ g^{bd} g_{e(a} C^e{}_{b)f} k^f C^a{}_{di} k^i &= \tfrac{1}{4} k^i k^f \nabla_f \gamma^{dk} \nabla_i \gamma_{dk} \ , \end{aligned}$$

together with

$$\begin{aligned} & -\gamma^{bd} \nabla_d k_e C^e{}_{bf} k^f - \tfrac{1}{2}\gamma^{ac} \nabla_e k_c C^e{}_{af} k^f - \tfrac{1}{2}\gamma_{ed} \nabla^d k^a C^e{}_{af} k^f = \\ & \qquad = -\tfrac{1}{2}\gamma^{bd} \nabla_d k^m k^f \nabla_f \gamma_{mb} - \tfrac{1}{2}\gamma^b{}_d \nabla^{(d} k^{m)} k^f \left(\nabla_b \gamma_{mf} + \nabla_f \gamma_{mb} - \nabla_m \gamma_{bf} \right) \ . \end{aligned}$$

By substitution of these relations in the former expression of the squared shear tensor, we finally obtain that

$$\begin{aligned} \hat{\sigma}_{ab} \hat{\sigma}^{ab}|_{\mathcal{H}^+} &= \tfrac{1}{2} \nabla^{(d} \xi^{c)} \nabla_d \xi_c + \nabla^m \xi^d k^i \nabla_i \gamma_{md} - \gamma_b{}^d \nabla^b \xi^c \nabla_d k_c - \gamma_b{}^d \nabla^c \xi^b \nabla_d k_c \\ & \quad - \tfrac{1}{2}\gamma^{bd} \nabla_d k^m k^f \nabla_f \gamma_{mb} - \tfrac{1}{2}\gamma^b{}_d \nabla^{(d} k^{m)} k^f \left(\nabla_b \gamma_{mf} + \nabla_f \gamma_{mb} - \nabla_m \gamma_{bf} \right) + \\ & - \gamma_a{}^c \gamma_c{}^g \nabla^a k^d \nabla_d k_g - \tfrac{1}{2}\gamma^c{}_a \gamma^a{}_e \nabla^d k^e \nabla_d k_c - \tfrac{1}{2}\gamma_{bd}\gamma_{ec} \nabla^b k^e \nabla^d k^c + \tfrac{1}{4} k^i k^f \nabla_f \gamma^{dk} \nabla_i \gamma_{dk} \end{aligned}$$

This result describes the relation between the squared shear tensor and the presence of quantum gravitational radiation, explicitly through $\boldsymbol{\gamma}$ and the geodesic correction $\boldsymbol{\xi}$, which both contribute to the right-hand side of the perturbed Raychaudhuri's equation (4.1).

From a physical point of view, the presence of the perturbation field has modified the background spacetime, deforming the shape of the congruence of radial null outgoing geodesics: at the initial time the event horizon of the Kruskal background is stable, since all the congruence parameters vanish on $\mathcal{H}^+$. As discussed in section 4.4, the presence of quantum gravitational radiation modifies this property, thus allowing for a positive flux of outgoing energy through $\mathcal{H}^+$, which here is accounted by $\hat{\sigma}_{ab}\hat{\sigma}^{ab}$. This situation is briefly sketched in figure 4.2, at the end of section 4.4.

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